

On a new harmonic heat flow with the reverse Hölder inequalities

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Abstract

This paper first proposes a new approximate scheme to construct a harmonic heat flow u between $(0, T) \times \mathbb{B}^d$ and $\mathbb{S}^D \subset \mathbb{R}^{D+1}$ with positive integers d and D : We agree that harmonic heat flow u means a solution of

$$\frac{\partial u}{\partial t} - \Delta u - |\nabla u|^2 u = 0.$$

It's scheme is crucially given by

$$\frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda + \lambda^{1-\kappa}(|u_\lambda|^2 - 1)u_\lambda = 0,$$

where the unknown mapping u_λ is from $(0, T) \times \mathbb{B}^d$ to \mathbb{R}^{D+1} with positive numbers λ and T , positive integers d and D and $\kappa(t) = \arctan(t)/\pi$ ($0 \leq t$).

The benefits to introduce a time-dependent parameter $\lambda^{1-\kappa}$ is readily to see

$$\int_{Q(T)} \lambda^{1-\kappa}(|u_\lambda|^2 - 1)^2 dz \leq C/\log \lambda$$

for some positive constant C independent of λ .

Next, making the best of it, we establish that a passing to the limits $\lambda \nearrow \infty$ (modulo subsequence of λ) brings the existence of a harmonic heat flow into spheres with

- (i) a global energy inequality,
- (ii) a monotonicity for the scaled energy,
- (iii) a reverse Poincaré inequality.

These inequalities (i), (ii) and (iii) improves the estimates on it's singular set contrast to the results by Y. Chen and M. Struwe [11], i.e. I show that a singular set of the new harmonic heat flow into spheres has at most finite $(d - \epsilon_0)$ -dimensional Hausdorff measure with respect to the parabolic metric whereupon ϵ_0 is a small positive number.

I believe that inequalities (i), (ii) and (iii) allow us to analyze how it behaves around its singularities.

1 Introduction.

Let \mathbb{B}^d and \mathbb{S}^D be respectively the unit ball centred at the origin in \mathbb{R}^d and the unit sphere in \mathbb{R}^{D+1} , where d and D are positive integers greater than or equal to 2. Consider the Sobolev space $H^{1,2}(\mathbb{B}^d; \mathbb{S}^D) := \{u \in H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}) ; |u| = 1 \text{ a.e. } x \in \mathbb{B}^d\}$. Giving a mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$, we say that the mapping $u \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$ is a weakly harmonic mapping (WHM) from \mathbb{B}^d into \mathbb{S}^D if the mapping u satisfies the following P.D.E in the weak sense:

$$\begin{cases} -\Delta u &= |\nabla u|^2 u & \text{in } \mathbb{B}^d, \\ u &= u_0 & \text{on } \partial\mathbb{B}^d. \end{cases} \quad (1.1)$$

Here “P.D.E in the weak sense” indicates that for $u - u_0 \in \mathring{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}) = \overline{C_0^\infty(\mathbb{B}^d; \mathbb{R}^{D+1})}^{H^{1,2}}$ and any $\phi \in C_0^\infty(\mathbb{B}^d; \mathbb{R}^{D+1})$,

$$\int_{\mathbb{B}^d} \langle \nabla u, \nabla \phi \rangle dx = \int_{\mathbb{B}^d} |\nabla u|^2 \langle u, \phi \rangle dx \quad (1.2)$$

holds. We employ the notation:

$$\begin{aligned} \nabla u &= \left(\frac{\partial u^i}{\partial x_\alpha} \right) \quad (\alpha = 1, \dots, d; i = 1, \dots, D+1), \quad \langle u, v \rangle = \sum_{i=1}^{D+1} u^i v^i, \\ \langle \nabla u, \nabla v \rangle &= \sum_{\alpha=1}^d \sum_{i=1}^{D+1} \frac{\partial u^i}{\partial x_\alpha} \frac{\partial v^i}{\partial x_\alpha}, \quad |\nabla u| = \sqrt{\langle \nabla u, \nabla u \rangle}. \end{aligned} \quad (1.3)$$

It is easy to check that (1.2) is the Euler-Lagrange equations for the following variational problem of minimizing the Dirichlet energy

$$I[w] = \int_{\mathbb{B}^d} |\nabla w|^2 dx$$

among mappings w belonging to the admissible function class

$$H_{u_0}^{1,2}(\mathbb{B}^d; \mathbb{S}^D) := \{w \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D); w - u_0 \in \mathring{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1})\}. \quad (1.4)$$

A work by R.Schoen and K.Uhlenbeck [30] can read that any minimizing mapping of I in $H_{u_0}^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$ denoted by u_{\min} are smooth except possibly a closed set having at most $(d-3)$ -Hausdorff dimension.

An alternative approach to find a harmonic map, “the heat flow method” introduced by J.Eells and J.H.Sampson [14] is now a standard one. They constructed a global harmonic heat flow from any compact manifold to any compact Riemannian manifold with non-positive sectional curvature.

Yielding to the approach, when we let a positive number T so large and a parabolic cylinder $Q(T)$ be $(0, T) \times \mathbb{B}^d$, we consider the heat flow:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u & \text{in } Q(T), \\ u(0, x) = u_0(x) & \text{at } \{0\} \times \mathbb{B}^d, \\ u(t, x) = u_0(x) & \text{on } [0, T) \times \partial \mathbb{B}^d. \end{cases} \quad (1.5)$$

We hereafter denote the time-slice mapping $v(t)$ on \mathbb{B}^d at a time t of a mapping $v(t, x)$ on $(0, T) \times \mathbb{B}^d$ by $v(t)(x) = v(t, x)$.

Define three function spaces:

$$\begin{aligned} L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)) \\ &:= \{u \mid u \text{ is measurable from } [0, T] \text{ to } H^{1,2}(\mathbb{B}^d; \mathbb{S}^D) \\ &\quad \text{with } \operatorname{esssup}_{t \in (0, T)} \|u(t)\|_{H^{1,2}(\mathbb{B}^d)} < +\infty\}, \\ H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1})) \\ &:= \left\{ u \mid u \text{ and } \partial u / \partial t \text{ (a weak derivative of } u) \right. \\ &\quad \left. \text{are second integrable from } [0, T] \text{ to } L^2(\mathbb{B}^d; \mathbb{R}^{D+1}) \right\}, \\ V(Q(T); \mathbb{S}^D) &= L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)) \cap H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1})). \end{aligned}$$

By virtue of topological obstruction, we generally have no hope about the existence of the classical solutions of the systems above (1.5). So we need a weak formulation of it. For any given mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$, we call a mapping $u \in V(Q(T); \mathbb{S}^D)$ a weakly harmonic heat flow (*WHHF*) provided for any $\phi \in C_0^\infty(Q(T); \mathbb{R}^{D+1})$

$$\int_{Q(T)} \left(\left\langle \frac{\partial u}{\partial t}, \phi \right\rangle + \langle \nabla u, \nabla \phi \rangle - \langle u, \phi \rangle |\nabla u|^2 \right) dz = 0, \quad (1.6)$$

$$u(t) - u_0 \in \mathring{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}) \quad \text{for almost every } t \in (0, T), \quad (1.7)$$

$$\lim_{t \searrow 0} u(t) = u_0 \quad \text{in } L^2(\mathbb{B}^d; \mathbb{R}^{D+1}). \quad (1.8)$$

Notice:

Theorem 1.1 (1.6) *is equivalent to*

$$\begin{cases} \frac{\partial u}{\partial t} \wedge u - \Delta u \wedge u = 0 & \text{in } (C_0^\infty(Q(T); \mathbb{R}^{D(D+1)/2}))^*, \\ |u| = 1 & \text{in a.e } z \in Q(T). \end{cases} \quad (1.9)$$

To construct some WHHF, Y.Chen [8] plied the following penalty scheme

$$\frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda + \lambda(|u_\lambda|^2 - 1)u_\lambda = 0 \quad (1.10)$$

and send $\lambda \nearrow \infty$. Y.Chen and M.Struwe [11] has established an existence and a partial regularity on a weakly harmonic heat flow between a compact Riemannian manifolds, and some later Y.Chen and F.H.Lin [9] generalized it the case from a compact Riemannian manifold with a boundary to a compact Riemannian manifold. Invoking a penalty approximation scheme, they proved that it is smooth except a set called “singular set” having at most the finite d -dimensional Hausdorff measure with respect to the parabolic metric. X.Cheng [12] showed that the time slice of singular set has at most $(d - 2)$ -dimensional Hausdorff measure at *every* time moment, instead of *almost* every time. Inspired by the work of L.Caffarelli, R.Kohn and L.Nirenberg [6] or L.C.Evans [16] or F.Hélein [22], Y.Chen, J.Li and F.H.Lin [10] and M.Feldman [18] discussed a partial regularity for a WHHF in a certain function class. Intrinsically, they assumed that their harmonic heat flow has a monotonicity for the scaled energy and a local energy estimate. Then they showed that such a flow possibly has a singular set with *zero* d -dimensional Hausdorff measure with respect to the parabolic metric.

As an improved approach to construct a WHHF, I will propose a new approximate evolutionary scheme said to be the Ginzburg-Landau heat flow and abbreviated by GLHF. To explain Ginzburg-Landau heat flow, we introduce smooth functions $\chi(t)$ and $\kappa(t)$ by

$$\begin{aligned} \chi(t) &= \begin{cases} t & (t < 2) \\ 3 & (t \geq 4), \end{cases} \quad \chi \leq 3, \\ \kappa(t) &= \arctan(t)/\pi. \end{aligned} \quad (1.11)$$

For a mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D) \cap H^{2,2}(\mathbb{B}^d \setminus B_{1-\delta_0}(0); \mathbb{S}^D)$ with a positive number sufficiently small δ_0 , a GLHF is designated by solution of the systems:

$$\frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda + \lambda^{1-\kappa} \dot{\chi}(|u_\lambda|^2 - 1)^2 (|u_\lambda|^2 - 1) u_\lambda = 0 \quad \text{in } Q(T), \quad (1.12)$$

$$u_\lambda = u_0 \quad \text{on } \partial Q(T). \quad (1.13)$$

If you notice that the non-linear term of $\lambda^{1-\kappa} \dot{\chi}(|u_\lambda|^2 - 1)^2 (|u_\lambda|^2 - 1) u_\lambda$ is bounded, Banach's fixed point theorem can state the unique existence of the mapping u_λ on $Q(T)$ with

- (a) $u_\lambda \in C^\infty(Q(T))$,
- (b) (1.12) is fulfilled in $Q(T)$,
- (c) $u_\lambda(t, x) - u_0(x) \in \mathring{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1})$ for almost every t in $(0, T)$,
- (d) $\lim_{t \searrow 0} \|u_\lambda(t, \circ) - u_0(\circ)\|_{L^2(\mathbb{B}^d)} = 0$.

We mention it in Theorem 2.1 in p.p 11. I emphasis that the time dependent parameter $\lambda^{1-\kappa}$ in (1.12) easily leads to $\int_{Q(T)} \lambda^{1-\kappa} (|u_\lambda|^2 - 1)^2 dz \leq C/\log \lambda$; This makes us handle with the nonlinear term. The reader should inquire Theorem 2.5.

We give a few remarks about the hypothesis on the mapping u_0 above. First of all, on account of a topological obstruction, a class of mappings $C^1(\overline{\mathbb{B}}^d; \mathbb{S}^{d-1})$ is empty if the degree of their restriction to the boundary doesn't vanish, whereas so isn't $H^{1,2}(\mathbb{B}^d; \mathbb{S}^{d-1})$ if d is more than or equal to 3. For instance, for any mapping $\phi \in C^1(\partial \mathbb{B}^d; \mathbb{S}^{d-1})$, the mapping $\phi(x/|x|)$ belongs to $H^{1,2}(\mathbb{B}^d; \mathbb{S}^{d-1})$ as long as d is more than or equal to 3. F.Betuel and X. Zheng [4] systematically studied a density result of various Sobolev mappings between two Riemannian manifolds. They pointed out that any map in $H^{1,2}(\mathbb{B}^d; \mathbb{S}^{d-1})$ is approximated by the mapping that is smooth except finite points. Second the further imposition of the initial and boundary mapping: the mapping u_0 belongs to $H^{2,2}$ near the boundary, is necessary for the legitimacy of Theorem 2.5 (Energy estimates), Theorem 2.6 (Energy decay estimates), Corollary 2.7 (Monotonicity for the scaled energy) and Theorem 2.12 (Parabolic Pokhojaev inequality).

The aim of the paper is to construct a WHHF with

- (i) a global energy inequality,
- (ii) a monotonicity for the scaled energy,

(iii) a reverse Poincaré inequality.

We should come to our mind that (ii) and (iii) are indispensable tools in a regularity or a partial regularity theory on the solutions of various elliptic and parabolic equations. This paper first establishes a crude bound, a maximal principle, a global energy inequality, a monotonicity inequality for the scaled energy and the reverse Poincaré inequality for GLHF; Next we show that the GLHF converges to a WHHF as $\lambda \nearrow \infty$ (modulo a subsequence of λ) and the flow also satisfies (i), (ii) and (iii). Thereafter by making the best of a monotonicity inequality for the scaled energy and the reverse Poincaré inequality, we will prove that the WHHF actually is smooth except on a small set called “**singular set.**” More precisely we assert

Theorem 1.2 (Partial Regularity). *Let d be a positive integer larger than or equal to 3. For a mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D) \cap H^{2,2}(\mathbb{B}^d \setminus B_{1-\delta_0}(0); \mathbb{S}^D)$ with a sufficiently small positive number δ_0 , there exists a WHHF and it is smooth on a certain open set in $Q(T)$ whose compliment has the finite $(d - \epsilon_0)$ -dimensional Hausdorff measure with respect to the parabolic metric, where ϵ_0 is a small positive number depending only on u_0 and d . The WHHF constructed above also holds*

$$(i) \quad \int_{T_1}^{T_2} dt \int_{\mathbb{B}^d} \left| \frac{\partial u}{\partial t} \right|^2 dx + \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u(T_2, \cdot)|^2 dx \leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u(T_1, \cdot)|^2 dx \quad (1.14)$$

for almost every time T_1 and T_2 with $0 \leq T_1 \leq T_2 \leq T$,

$$\begin{aligned} (ii) \quad & \int_{R_1}^{R_2} \frac{d\rho}{\rho^{d-1}} \int_{t_0-(2\rho)^2}^{t_0-\rho^2} dt \int_{\mathbb{B}^d} \left| \frac{\partial u}{\partial t} + \frac{x - x_0}{2(t - t_0)} \cdot \nabla u \right|^2 \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right) dx \\ & + \frac{1}{2R_1^d} \int_{t_0-(2R_1)^2}^{t_0-R_1^2} dt \int_{\mathbb{B}^d} |\nabla u|^2 \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right) dx \\ & \leq \frac{1}{2R_2^d} \int_{t_0-(2R_2)^2}^{t_0-R_2^2} dt \int_{\mathbb{B}^d} |\nabla u|^2 \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right) dx + \frac{C_M}{2}(R_2^2 - R_1^2) \end{aligned} \quad (1.15)$$

for any positive numbers R_1 and R_2 with $R_1 \leq R_2$ and an arbitrary point $z_0 = (t_0, x_0)$ in $Q(T)$ satisfying $t_0 - (2R_2)^2 > 0$, where C_M is a certain positive constant depending only on d , z_0 and u_0 , and in addition

$$(iii) \quad \int_{P_R(z_0)} |\nabla u(z)|^2 dz \leq \frac{C}{R^2} \int_{P_{2R}(z_0)} |u(z) - a(t)|^2 dz \quad (1.16)$$

for any t -variable second integrable mapping $a = a(t) = (a^i(t))$ ($i = 1, 2, \dots, D+1$) and any parabolic cylinder $P_{2R}(z_0)$ compactly contained in $Q(T)$.

We next discuss an asymptotic behaviour of the WHHF as $t \nearrow \infty$; The final part of the paper demonstrates that our WHHF with a constant boundary value converges strongly to a constant as $t \nearrow \infty$ in $H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$ as long as the dimension d is more than or equal to 3. A surprising example by K-C. Chang and W-Y. Ding [7] tells us that the same result doesn't hold in $d = 2$. They constructed a smooth harmonic heat flow $u = u(t, x)$ under a smooth initial condition u_0 with $u_0(\mathbb{B}^2) = \mathbb{S}^2$ and a boundary condition $u_0|_{\partial\mathbb{B}^2} = a \text{ constant}$, and showed that it *does not* converge to any WHM in $C^0(\mathbb{B}^2)$ as $t \nearrow \infty$.

Their result entails that $u(t)(\mathbb{B}^2) = \mathbb{S}^2$ holds at each time t in $(0, \infty)$. On the other hand, non-existence result by L.Lemaire [24] and [25] stands for

Theorem 1.3 *Let a mapping $\phi : \mathbb{B}^2 \longrightarrow \mathbb{S}^2$ be harmonic with $\phi|_{\partial\mathbb{B}^2} = a \text{ constant}$. Then the mapping ϕ must be the same constant.*

Thus the harmonic heat flow constructed above is not homotopic to ϕ nevertheless it has the same boundary condition. So we can not expect that the harmonic heat flow u converges to *a constant* as $t \nearrow \infty$ in $C^0(\mathbb{B}^2)$. In a nut shell the square sum of the gradients of the harmonic heat flow by K-C. Chang and W-Y. Ding, converges to the Dirac measure as $t \nearrow \infty$. When $d = 2$, we should notice that the WHHF is unique due to A. [19] under the following criterion:

$$t \longmapsto \int_{\mathbb{B}^d} |\nabla u(t)|^2 dx \quad (1.17)$$

is non-increasing in $[0, \infty)$. On the other hand, we know the existence of infinite many WHHF's not satisfying the criterion above. This is by M.Bertsch, R.Dal Passo and R. Vd Hout [1].

When d is greater than or equal to 3, the equator map $x/|x| \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^{d-1})$ ¹ reveal us the WHHF possessing a singularity. Thus any WHHF may break any given topology. In $d = 3$, the uniqueness of WHHF fails even if (1.17) does hold and u_0 is a WHM by J.M.Coron [13] or any stationary by M.-C.Hong [21] or a minimizer by M.Bertsch, R. Dal Passo and A. Pisante [2].

Our alternative main theorem is

Theorem 1.4 *Let a mapping u be the WHHF obtained by the limit of GLHF as $\lambda \nearrow \infty$ (modulo of subsequence of λ). Then we can extend the WHHF*

¹This map is also WHM and even the absolute minimizer. We refer it to H. Brezis, J.M.Coron and E.H.Lieb [5]

from $Q(T)$ to $Q(\infty)$ and moreover the mapping $u(t)$ converges strongly to a constant in $H^{1,2}(\mathbb{B}^d)$ as $t \nearrow \infty$ if $u(t)|_{\partial\mathbb{B}^d} = \text{the constant}$ at almost all $t > 0$.

This result can be regarded as a parabolic analogue of non-existence result of harmonic mappings. The proofs of Theorem 1.2 and Theorem 1.4 will be given in the final part of the paper.

We close this introduction by enumerating a glossary of notation:

Notation

- (i) $\mathbb{B}^d = \{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d; |x| = \sqrt{\sum_{\alpha=1}^d (x_\alpha)^2} < 1\}$
and $\partial\mathbb{B}^d$ is the boundary of \mathbb{B}^d .
- (ii) $C^d = \{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d; |x_\alpha| < 1 \ (\alpha = 1, \dots, d)\}$.
- (iii) $Q(T) = (0, T) \times \mathbb{B}^d$. $\partial Q(T) = [0, T] \times \partial\mathbb{B}^d \cup \{0\} \times \mathbb{B}^d$.
- (iv) $\mathbb{S}^D = \{y = (y^1, y^2, \dots, y^{D+1}) \in \mathbb{R}^{D+1}; |y| = \sqrt{\sum_{i=1}^{D+1} (y^i)^2} = 1\}$.
- (v) For any points $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$, the parabolic metric $d(z_1, z_2)$ means $|t_1 - t_2|^{1/2} + |x_1 - x_2|$.
- (vi) For any positive integer n , $Q_n = [1/n, T - 1/n] \times \overline{B_{1-1/n}(0)}$ and $d_n = d(Q_n, \partial Q(T))$.
- (vii) For a Lebesgue measurable subset A in \mathbb{R}^d or $Q(T)$, $|A|$ denotes the d or $(d+2)$ -dimensional Lebesgue measure of A .
- (viii) Set points $x = (x_1, x_2, \dots, x_d)$, $x_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,d})$ and $z_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$; Then indicate a ball $B_r(x_0)$, a cube $C_r(x_0)$, a parabolic cylinder $P_r(z_0)$ and a parabolic cube $D_r(z_0)$ by

$$\begin{aligned}
B_r(x_0) &= \{x \in \mathbb{R}^d; |x - x_0| < r\}, \\
C_r(x_0) &= \{x \in \mathbb{R}^d; |x_i - x_{0,i}| < r, \ (i = 1, 2, \dots, d)\}, \\
P_r(z_0) &= \{z = (t, x) \in Q(T); t_0 - r^2 < t < t_0 + r^2, |x - x_0| < r\}, \\
D_r(z_0) &= (t_0 - r^2, t_0 + r^2) \times C_r(x_0).
\end{aligned}$$

In $B_r(x_0)$, $C_r(x_0)$, $P_r(z_0)$ and $D_r(z_0)$, the points of x_0 and z_0 will be often abbreviated when no confusion may arise.

- (ix) For a set $A \subset \mathbb{R}^d$ and $0 \leq d' < \infty$, we define the d' -dimensional Hausdorff measure ² with respect to the parabolic metric by

$$\mathcal{H}^{(d')}(A) = \lim_{R \searrow 0} \mathcal{H}_R^{(d')}(A)$$

$$\text{with } \mathcal{H}_R^{(d')}(A) := \inf_{\text{covering}} \left\{ \sum_i R_i^{d'}; A \subset \bigcup_i P_{R_i}(z_i), 0 < R_i < R \right\}.$$

- (x) Letter C denotes a generic constant. By the letter $C(B)$, it means that a constant depends only on a parameter B .
- (xi) $\kappa(t)$ is $\arctan(t)/\pi$.
- (xii) $[*]$ is the Gauss symbol.

Functions and Derivatives

- (i) For vectors u and $v \in \mathbb{R}^{D+1}$, $\langle u, v \rangle = \sum_{i=1}^{D+1} u^i v^i$ and $|u| = \langle u, u \rangle^{1/2}$.
- (ii) For a map $v : \mathbb{R}^d \rightarrow \mathbb{R}^{D+1}$ the gradient matrix of v is defined by $\nabla v = (\partial v^i / \partial x_\alpha)$ ($\alpha = 1, \dots, d; i = 1, \dots, D+1$), while $\partial_t v$ is by $\partial v / \partial t$. In addition, for $x = (x_1, x_2, \dots, x_d)$, $x \cdot \nabla$ and ∇_ν respectively denotes $\sum_{\alpha=1}^d x_\alpha \partial / \partial x_\alpha$ and $\nu \cdot \nabla$, a symbol ν denoting the outward normal unit vector on the boundary of a discussed domain.

On the contrary, differentials $\nabla_\tau = (\nabla_{\tau_i})$ ($i = 1, 2, \dots, d-1$) indicates by $\nabla - \nu \nu \cdot \nabla$ and $\triangle_\tau = \sum_{j=1}^{d-1} \nabla_{\tau_j}^2$.

For any positive integer k , a higher differentials ∇^k and ∇_τ^k mean

$$\nabla^k = \sum_{\substack{k_1 + \dots + k_{d-1} = k \\ 0 \leq k_1, \dots, k_{d-1} \leq k}} \nabla_{x_1}^{k_1} \dots \nabla_{x_{d-1}}^{k_{d-1}},$$

$$\nabla_\tau^k = \sum_{\substack{k_1 + \dots + k_{d-1} = k \\ 0 \leq k_1, \dots, k_{d-1} \leq k}} \nabla_{\tau_1}^{k_1} \dots \nabla_{\tau_{d-1}}^{k_{d-1}}$$

- (iii) For the gradient matrices of u and v , namely ∇u and ∇v , we mean $\langle \nabla u, \nabla v \rangle = \sum_{\alpha=1}^d \sum_{i=1}^{D+1} \nabla_\alpha u^i \nabla_\alpha v^i$ and $|\nabla u|^{1/2} = \langle \nabla u, \nabla u \rangle^{1/2}$.
- (iv) For GLHF u_λ , we call \mathbf{e}_λ the Ginzburg-Landau energy density given by $|\nabla u_\lambda|^2/2 + \lambda^{1-\kappa} \chi((|u_\lambda|^2 - 1)^2)/4$.

²This definition is not it on the usual Hausdorff measure. However it is sufficient to our end.

- (v) For a mapping $u(t, x)$ on $(t, x) \in Q(T)$, a time-slice mapping $u(t)$ at a time t , denotes $u(t)(x) = u(t, x)$.

Function spaces

- (i) What a function f belongs to $C^0(\mathbb{B}^d)$ or $C^0(Q(T))$ is that the function f is continuous on \mathbb{B}^d or $Q(T)$.
- (ii) $C_0^\infty(\mathbb{B}^d)$ or $C_0^\infty(Q(T))$ is respectively the space of infinite differentiable function with a compact support in \mathbb{B}^d or $Q(T)$ and $(C_0^\infty)^*$ is the dual space of it.
- (iii) We say that a function f belongs to Hölder space on $\overline{Q(T)}$ if there is a positive constant C and a positive number α_0 ($0 < \alpha_0 < 1$) such that

$$|f(z_1) - f(z_2)| \leq C d(z_1, z_2)^{\alpha_0} \quad \text{for any } z_1, z_2 \in \overline{Q(T)}.$$

The semi-norm of such a function f is given by

$$[f]_{C^{\alpha_0}(\overline{Q(T)})} = \sup_{\substack{z_1, z_2 \in \overline{Q(T)}, \\ z_1 \neq z_2}} \frac{|f(z_1) - f(z_2)|}{d(z_1, z_2)^{\alpha_0}}.$$

- (iv) $C^{2, \alpha_0}(\overline{Q(T)}) = \{f \in C^0(\overline{Q(T)}) ; \nabla f, \nabla^2 f \text{ and } \partial_t f \text{ is continuous on } \overline{Q(T)}\}$ of which the norm

$$\begin{aligned} \|f\|_{C^{2, \alpha_0}(\overline{Q(T)})} &= \sup_{z \in \overline{Q(T)}} |f(z)| + \sup_{z \in \overline{Q(T)}} |\nabla f(z)| \\ &\quad + \sup_{z \in \overline{Q(T)}} |\nabla^2 f(z)| + \sup_{z \in \overline{Q(T)}} |\partial f / \partial t(z)| \\ &\quad + [\nabla^2 f]_{C^{\alpha_0}(\overline{Q(T)})} + [\partial f / \partial t]_{C^{\alpha_0/2}(\overline{Q(T)})} \end{aligned}$$

is finite.

- (v) $L^p(\mathbb{B}^d)$ or $L^p(Q(T))$ respectively means the space of the p th summable function on \mathbb{B}^d or $Q(T)$ with the norm of $\|f\|_{L^p(\mathbb{B}^d)} = \left(\int_{\mathbb{B}^d} |f|^p dx \right)^{1/p}$ or $\|f\|_{L^p(Q(T))} = \left(\int_{Q(T)} |f|^p dz \right)^{1/p}$. On the contrary, $L^\infty(\mathbb{B}^d(Q(T)))$ is the space of any summable function so that the norm of $\|f\|_{L^\infty(\mathbb{B}^d)}$ is $\sup_{x \in \mathbb{B}^d} |f(x)|$ and the one of $\|f\|_{L^\infty(Q(T))}$ is $\sup_{z \in Q(T)} |f(z)|$.
- (vi) $H^{1,2}(\mathbb{B}^d) = \{f \in L^2(\mathbb{B}^d) ; \partial f / \partial x_\alpha \in L^2(\mathbb{B}^d), (\alpha = 1, \dots, d)\}$.

- (vii) $H^{1,2}(Q(T)) = \{f \in L^2(Q(T)); \partial f / \partial x_\alpha, \partial_t f \in L^2(Q(T)), (\alpha = 1, \dots, d)\}.$
- (viii) $H^{2,p}(Q(T)) = \{f \in L^p(Q(T)); \partial f / \partial x_\alpha, \partial^2 f / \partial x_\alpha \partial x_\beta, \partial_t f \in L^p(Q(T)), (\alpha, \beta = 1, \dots, d)\},$ where a number p is more than or equal to 1.

In the following, let $X(\mathbb{B}^d(\text{or } Q(T)))$ be a Banach space on $\mathbb{B}^d(\text{or } Q(T)).$

- (ix) If a function f belongs to $X_{\text{loc}}(\mathbb{B}^d(\text{or } Q(T)))$, this means that the function f is of $X_{\text{loc}}(\Omega)$ for any set Ω compactly contained in $\mathbb{B}^d(\text{or } Q(T)).$
- (ix) $\overset{\circ}{X}(\mathbb{B}^d)$ is the subspace of $X(\mathbb{B}^d)$ whose element vanishes on $\partial\mathbb{B}^d$ in the trace sense.
- (x) $X(\mathbb{B}^d(\text{or } Q(T)); \mathbb{R}^{D+1}) = \{u = (u^i) : \mathbb{B}^d(\text{or } Q(T)) \rightarrow \mathbb{R}^{D+1} ; u^i \in X(\mathbb{B}^d(\text{or } Q(T))), (i = 1, \dots, D+1)\}.$
- (xi) $X(\mathbb{B}^d(\text{or } Q(T)); \mathbb{S}^D) = \{u = (u^i) \in X(\mathbb{B}^d(\text{or } Q(T)); \mathbb{R}^{D+1}); |u| = 1 \text{ a.e } x \in \mathbb{B}^d \text{ (or a.e } z \in Q(T)), (i = 1, \dots, D+1)\}.$
- (xii) If we say that a mapping $u = u(t, x)$ on $(0, T) \times \mathbb{B}^d$ belongs to $Y(0, T; X(\mathbb{B}^d; \mathbb{R}^{D+1}(\text{or } \mathbb{S}^D)))$, it means $u(t) \in X$ and $\|u(t)\|_X \in Y$, where a symbol $\|\cdot\|_X$ is the equipped norm in a normed space X .
- (xiii) Let a mapping v be any mappings belonging to $H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$. If a mapping $w \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$ and $w - v \in \overset{\circ}{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1})$, we then call the mapping w belong to $H_v^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$.
- (xiv) $V(Q(T); \mathbb{S}^D) = L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)) \cap H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1})).$

2 GLHF.

The chapter is devoted to the study of GLHF. We prove that a GLHF satisfies a crude bound, a maximal principle, a few energy inequalities, a monotonicity inequality for the scaled energy and finally a hybrid type inequality.

2.1 Properties on GLHF.

As alluded in Chapter 1, we briefly discuss how to construct the GLHF. We first assume that the initial and boundary mapping u_0 is smooth in $\overline{Q(T)}$ in Theorem 2.1, Lemma 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6, Corollary 2.7 and Theorem 2.8: We state

Theorem 2.1 (The existence of GLHF). *Give a smooth mapping u_0 between $\overline{Q(T)}$ and \mathbb{R}^{D+1} . Then there exists the classical solution $u_\lambda \in C^\infty(\overline{Q(T)})$ to (1.12) under $u_\lambda = u_0$ on $\partial Q(T)$ such that when we set $u_\lambda^{(1)}$ and $u_\lambda^{(2)}$ respectively the classical solution to (1.12) with the smooth boundary and initial condition $u_0^{(1)}$ and $u_0^{(2)}$ on $\partial Q(T)$, we have*

$$\begin{aligned} & \|u_\lambda^{(1)} - u_\lambda^{(2)}\|_{H^{2,2}(Q(T))} \\ & \leq C(\lambda, T)(1 + \|u_0^{(1)} - u_0^{(2)}\|_{H^{1,2}(\mathbb{B}^d)} + \|u_0^{(1)} - u_0^{(2)}\|_{H^{2,2}(\mathbb{B}^d \setminus B_{1-\delta_0}(0))}) \end{aligned} \quad (2.1)$$

with a positive number δ_0 sufficiently small.

Proof of Theorem 2.1.

A routine work by means of a Duhamel's formula and a contraction mapping theorem implies the unique classical solution to (1.12) on $Q(t_\lambda)$ for a small positive number t_λ possibly depending on λ . We repeat the argument above to extend our solution to the time interval $[t_\lambda, 2t_\lambda]$. Continuing after finite steps, we eventually come up with the classical solution to (1.12) on $Q(T)$.

Moreover, since the mapping $u_\lambda^{(1)} - u_\lambda^{(2)}$ is the solution to

$$\begin{cases} \frac{\partial(u_\lambda^{(1)} - u_\lambda^{(2)})}{\partial t} - \Delta(u_\lambda^{(1)} - u_\lambda^{(2)}) \\ + \lambda^{1-\kappa} [\dot{\chi}((|u_\lambda^{(1)}|^2 - 1)^2)(|u_\lambda^{(1)}|^2 - 1)u_\lambda^{(1)} - \dot{\chi}((|u_\lambda^{(2)}|^2 - 1)^2)(|u_\lambda^{(2)}|^2 - 1)u_\lambda^{(2)}] \\ = 0 \quad \text{in } Q(T), \\ u_\lambda^{(1)} - u_\lambda^{(2)} = u_0^{(1)} - u_0^{(2)} \quad \text{on } \partial Q(T), \end{cases} \quad (2.2)$$

by applying Theorem 9.1 in Ladyžhenskaya, O. A., Solonnikov, V. A., Ural'ceva, N. N. [23, p.341] to (2.2), we assert (2.1). \square

Remark 2.2 *We call the classical solution in Theorem 2.1 “the classical GLHF.”*

Next we introduce a crude bound:

Lemma 2.3 (Crude Bound). *A parabolic analogue to Bethuel, F., Brezis, H. and Hélein, R. [3, Lemma A.1] tells us that*

$$\begin{aligned} \|\nabla u_\lambda\|_{L_{\text{loc}}^\infty(Q(T))} & \leq \frac{C}{\sqrt{\lambda}}, \quad \|\nabla^2 u_\lambda\|_{L_{\text{loc}}^\infty(Q(T))} \leq \frac{C}{\lambda}, \\ \|\nabla^3 u_\lambda\|_{L_{\text{loc}}^\infty(Q(T))} & \leq \frac{C}{\lambda\sqrt{\lambda}}, \quad \|\nabla^4 u_\lambda\|_{L_{\text{loc}}^\infty(Q(T))} \leq \frac{C}{\lambda^2} \end{aligned} \quad (2.3)$$

hold for the classical GLHF.

Next we prove a maximal principle:

Theorem 2.4 (Maximal Principle). *Each of the classical GLHF $\{u_\lambda\}$ ($\lambda > 0$) satisfies*

$$|u_\lambda| \leq 1 \quad \text{for any point } z \in Q(T). \quad (2.4)$$

Proof of Theorem 2.4.

Set the truncation function $(|u_\lambda|^2 - 1)^{(0)}$ as

$$(|u_\lambda|^2 - 1)^{(0)} = \begin{cases} 0 & (|u_\lambda|^2 \leq 1), \\ |u_\lambda|^2 - 1 & (|u_\lambda|^2 > 1). \end{cases}$$

A multiplier of (1.12) by $(|u_\lambda|^2 - 1)^{(0)}u_\lambda$ and integrate it on $(0, t) \times \mathbb{B}^d$ with any t in $(0, T)$ observes

$$\begin{aligned} & \int_0^t dt \int_{\mathbb{B}^d} \left\langle \frac{\partial u_\lambda}{\partial t}, u_\lambda \right\rangle (|u_\lambda|^2 - 1)^{(0)} dx \\ & + \int_0^t dt \int_{\mathbb{B}^d} |\nabla u_\lambda|^2 (|u_\lambda|^2 - 1)^{(0)} dx + \frac{1}{2} \int_0^t dt \int_{\mathbb{B}^d} |\nabla (|u_\lambda|^2 - 1)^{(0)}|^2 dx \\ & + \int_0^t \lambda^{1-\kappa} dt \int_{\{x \in \mathbb{B}^d; |u_\lambda| \geq 1\}} \dot{\chi}((|u_\lambda|^2 - 1)^2) (|u_\lambda|^2 - 1) (|u_\lambda|^2 - 1)^{(0)} |u_\lambda|^2 dx = 0. \end{aligned} \quad (2.5)$$

Note that

$$\int_0^t dt \int_{\mathbb{B}^d} \left\langle \frac{\partial u_\lambda}{\partial t}, u_\lambda \right\rangle (|u_\lambda|^2 - 1)^{(0)} dx = \frac{1}{4} \int_0^t dt \frac{d}{dt} \int_{\mathbb{B}^d} ((|u_\lambda|^2 - 1)^{(0)})^2 dx.$$

Since the second, the third and the fourth terms in (2.5) are nonnegative, we thus infer

$$\int_0^t dt \frac{d}{dt} \int_{\mathbb{B}^d} ((|u_\lambda|^2 - 1)^{(0)})^2 dx \leq 0. \quad (2.6)$$

So we arrive at

$$\int_{\mathbb{B}^d} ((|u_\lambda(t)|^2 - 1)^{(0)})^2 dx \leq \int_{\mathbb{B}^d} ((|u_0|^2 - 1)^{(0)})^2 dx,$$

which can read $|u_\lambda| \leq 1$ at almost all $z \in Q(T)$. Since the mapping u_λ is continuous, we claim (2.4). \square

We mention three fundamental energy inequalities. We only mimic those for the usual linear heat flow: Then a multiplier of (1.12) by $\partial u_\lambda / \partial t$, an integration of it on $Q(T)$ permit us to state

Theorem 2.5 (Energy Estimate). *For any numbers t_1 and t_2 with $0 \leq t_1 \leq t_2 \leq T$, the classical GLHF u_λ satisfies*

$$\begin{aligned} & \int_{t_1}^{t_2} dt \int_{\mathbb{B}^d} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 dx + \int_{\mathbb{B}^d} \mathbf{e}_\lambda(t_2, x) dx \\ & + \frac{\log \lambda}{4} \int_{t_1}^{t_2} \dot{\kappa} \lambda^{1-\kappa} dt \int_{\mathbb{B}^d} \chi(|u_\lambda|^2 - 1)^2 dx = \int_{\mathbb{B}^d} \mathbf{e}_\lambda(t_1, x) dx. \end{aligned} \quad (2.7)$$

The third theorem is used for the proof of Theorem 1.4:

Theorem 2.6 (Energy Decay Estimate). *Assume $u_0|_{\partial \mathbb{B}^d} = a$ constant. Then the following*

$$\int_{\mathbb{B}^d} \mathbf{e}_\lambda(|x|^2 + 1) dx e^{(d-2)t} \leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u_0|^2 (|x|^2 + 1) dx \quad (2.8)$$

is valid for the classical GLHF and in any time $t \in (0, T)$.

Proof of Theorem 2.6.

Multiply (1.12) by $(\partial u_\lambda / \partial t)(|x|^2 + 1)$ and integrate it over \mathbb{B}^d to verify

$$\begin{aligned} & \int_{\mathbb{B}^d} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 (|x|^2 + 1) dx + \frac{d}{dt} \int_{\mathbb{B}^d} \mathbf{e}_\lambda(|x|^2 + 1) dx \\ & + 2 \int_{\mathbb{B}^d} \left\langle x \cdot \nabla u_\lambda, \frac{\partial u_\lambda}{\partial t} \right\rangle dx + \log \lambda \frac{\dot{\kappa} \lambda^{1-\kappa}}{4} \int_{\mathbb{B}^d} \chi(|u_\lambda|^2 - 1)^2 (|x|^2 + 1) dx = 0. \end{aligned} \quad (2.9)$$

Next, a multiplier of (1.12) by $-2x \cdot \nabla u_\lambda$ and an integration of it over \mathbb{B}^d imply

$$-2 \int_{\mathbb{B}^d} \left\langle \frac{\partial u_\lambda}{\partial t}, x \cdot \nabla u_\lambda \right\rangle dx + 2(d-2) \int_{\mathbb{B}^d} \mathbf{e}_\lambda dx \quad (2.10)$$

$$+ \int_{\partial \mathbb{B}^d} \left(\left| \frac{\partial u_\lambda}{\partial |x|} \right|^2 - |\nabla_\tau u_\lambda|^2 \right) d\mathcal{H}_x^{d-1} \leq 0.$$

Summing up (2.9) and (2.10), noting $u_0|_{\partial \mathbb{B}^d} = a \text{ constant}$ in the trace sense and multiplying it by $e^{(d-2)t}$, we arrive at

$$\frac{d}{dt} \left(\int_{\mathbb{B}^d} \mathbf{e}_\lambda(|x|^2 + 1) dx e^{(d-2)t} \right) \leq 0, \quad (2.11)$$

which concludes our result by integrating from 0 to any positive number $t \in (0, T)$ with respect to t . \square

By combining Theorem 2.5 with the proof of Theorem 2.6, we obtain the following inequality. We call it ‘‘A parabolic Pokhojaev inequality’’.

Corollary 2.7 (Parabolic Pokhojaev Inequality). *Let the mapping u_λ be the classical GLHF; We infer*

$$\begin{aligned} & \int_0^T dt \int_{\partial \mathbb{B}^d} \left| \frac{\partial u_\lambda}{\partial |x|} \right|^2 d\mathcal{H}_x^{d-1} \\ & \leq T \int_{\partial \mathbb{B}^d} |\nabla_\tau u_0|^2 d\mathcal{H}_x^{d-1} + \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u_0|^2 (|x|^2 + 1) dx. \end{aligned} \quad (2.12)$$

We finally introduce a local energy inequality without a proof. It will be used a several times in the rest of the paper.

Theorem 2.8 (Local Energy Inequality). *The following inequality*

$$\begin{aligned} & \int_{P_R(z_0)} \left| \frac{\partial u_\lambda}{\partial t}(z) \right|^2 dz + \operatorname{esssup}_{t_0 - R^2 < t < t_0 + R^2} \int_{B_R(z_0)} \mathbf{e}_\lambda(t, x) dx \\ & \leq \frac{C}{R^2} \int_{P_{2R}(z_0)} \mathbf{e}_\lambda(z) dz \end{aligned} \quad (2.13)$$

holds for the classical GLHF and any parabolic cylinder $P_{2R}(z_0)$ compactly contained in $Q(T)$.

Theorem 2.1, Lemma 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6, Corollary 2.7 and Theorem 2.8 are valid for less stringent smoothness requirement for the initial and boundary condition u_0 belonging to $H^{1,2}(\mathbb{B}^d; \mathbb{S}^D) \cap H^{2,2}(\mathbb{B}^d \setminus B_{1-\delta_0}(0); \mathbb{S}^D)$ with a positive number sufficiently small δ_0 : Indeed, take the mollifier of the mapping u_0 and passing to the limit, we readily see

Theorem 2.9 (GLHF). *Give a mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D) \cap H^{2,2}(\mathbb{B}^d \setminus B_{1-\delta_0}(0); \mathbb{S}^D)$ with a positive number δ_0 sufficiently small, the GLHF u_λ , i.e. the mapping (a),(b),(c),(d) in p.p 5 exists. In addition, the GLHF satisfies Lemma 2.3, Theorem 2.4, Theorem 2.5 Theorem 2.6, Corollary 2.7 and Theorem 2.8.*

Remark 2.10 *Since $\dot{\chi}(|u_\lambda|^2 - 1)^2 = 1$ in $0 \leq |u_\lambda| \leq 1$, (1.12) and \mathbf{e}_λ reduce to*

$$\frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda + \lambda^{1-\kappa}(|u_\lambda|^2 - 1)u_\lambda = 0, \quad (2.14)$$

$$\frac{1}{2}|\nabla u_\lambda|^2 + \frac{\lambda^{1-\kappa}}{4}\chi(|u_\lambda|^2 - 1)^2. \quad (2.15)$$

Remark 2.11 *Henceforth, if we quote to the one of inequalities above, it means the one for the GLHF subject to the initial and boundary mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D) \cap H^{2,2}(\mathbb{B}^d \setminus B_{1-\delta_0}(0); \mathbb{S}^D)$ with a positive number sufficiently small δ_0 . For instance, if we say “from (2.4) in Theorem 2.4,” it indicates “from (2.4) in Theorem 2.4 which holds for the GLHF above,”*

2.2 Monotonicity For Scaled Energy.

We introduce a monotonicity inequality for the scaled energy. In our settings, see Y.Chen and F.H.Lin [9] about it's proof. Likewise Theorem 2.5 etc, approximate smoothly the boundary condition, note (2.12) of Corollary 2.7 and pass to the limit to read

Theorem 2.12 (Monotonicity for Scaled Energy). *For any point $z_0 = (t_0, x_0) \in Q(T)$ and any positive number R with $t_0 - (2R)^2 > 0$, the scaled energy is denoted by*

$$E_\lambda(R; z_0) = \frac{1}{R^d} \int_{t_0-(2R)^2}^{t_0-R^2} dt \int_{\mathbb{B}^d} \mathbf{e}_\lambda \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) dx. \quad (2.16)$$

Then we have

$$\begin{aligned} \frac{dE_\lambda}{dR}(R; z_0) \geq & \quad (2.17) \\ - \frac{1}{R^{d-1}} \int_{t_0-(2R)^2}^{t_0-R^2} \frac{t-t_0}{R^2} dt \int_{\mathbb{B}^d} & \left| \frac{\partial u_\lambda}{\partial t} + \frac{x-x_0}{2(t-t_0)} \cdot \nabla u_\lambda \right|^2 \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) dx \end{aligned}$$

$$+ \frac{1}{2R^{d+1}} \int_{t_0-(2R)^2}^{t_0-R^2} \lambda^{1-\kappa} \int_{\mathbb{B}^d} (|u_\lambda|^2 - 1)^2 \exp\left(\frac{|x - x_0|}{4(t - t_0)}\right) dx - C_M R$$

$$\text{with } d_0 = d(x_0, \partial\mathbb{B}^d) \text{ and } C_M = \frac{C}{d_0^{d+2}} \left(\int_{\partial\mathbb{B}^d} |\nabla_\tau u_0|^2 d\mathcal{H}_x^{d-1} + \int_{\mathbb{B}^d} |\nabla u_0|^2 dx \right).$$

Corollary 2.13 *An integration of (2.17) from R_1 to R_2 with $0 < R_1 < R_2$ over R yields*

$$\begin{aligned} E_\lambda(R_2; z_0) &\geq E_\lambda(R_1; z_0) \\ &+ \int_{R_1}^{R_2} \frac{dR}{R^{d-1}} \int_{t_0-(2R)^2}^{t_0-R^2} dt \int_{\mathbb{B}^d} \left| \frac{\partial u_\lambda}{\partial t} + \frac{x - x_0}{2(t - t_0)} \cdot \nabla u_\lambda \right|^2 \exp\left(\frac{|x - x_0|}{4(t - t_0)}\right) dx \\ &- \frac{C_M}{2} (R_2^2 - R_1^2). \end{aligned} \quad (2.18)$$

2.3 Hybrid type Inequality for GLHF.

We are in the position to prove an inequality of the hybrid type; This inequality is the one of the crucial tools in the paper. We claim

Theorem 2.14 (Hybrid Inequality). *For any positive number ϵ_0 , any point z_0 in $Q(T)$, setting d_0 as $d_0 = \text{dist}(z_0, \partial Q(T))$, there exists a positive constant $C(\epsilon_0, d_0)$ satisfying $C(\epsilon_0, d_0) \nearrow \infty$ as $\epsilon_0 \searrow 0$ or $d_0 \searrow 0$ such that*

$$\begin{aligned} \int_{P_R(z_0)} \mathbf{e}_\lambda(z) dz &\leq \epsilon_0 \int_{P_{2R}(z_0)} \mathbf{e}_\lambda(z) dz + \frac{C(\epsilon_0, d_0)}{R^2} \int_{P_{2R}(z_0)} |u_\lambda(z) - a(t)|^2 dz \\ &+ \frac{C(\epsilon_0, R, d_0)}{\log \lambda}, \end{aligned} \quad (2.19)$$

where $P_{2R}(z_0)$ is any parabolic cylinder compactly contained in $Q(T)$ and $a = a(t) = (a^i(t))$ ($i = 1, 2, \dots, D + 1$) is any L^2 -mapping with respect to a positive parameter t .

As a preliminary we list symbols and auxiliary functions employed only here. Give L_λ by $[\lambda^{3/(1+\theta_0)}]$. We then first introduce the decomposition convention: Put

$$\Delta\theta_l = (1/2)^l, \quad \Delta r_l = \Delta\rho_l = r(1/2)^l \quad (l = 1, 2, \dots, L_\lambda),$$

$$\rho_l = \begin{cases} 0 & (l = 0) \\ (1 - \epsilon_0^4)r & (l = 1) \\ (1 - \epsilon_0^4)r + C_1 \epsilon_0^4 r \sum_{j=1}^{l-1} \Delta \theta_j & (l = 2, \dots, L_\lambda) \end{cases}$$

with $C_1 = \left(\sum_{l=1}^{L_\lambda-1} \Delta \theta_l \right)^{-1}$,

$$\begin{cases} \rho_l^b = \rho_l - 2\Delta \rho_l / 3 \\ \rho_l^f = \rho_l - \Delta \rho_l / 3 \end{cases} \quad (l = 1, 2, \dots, L_\lambda).$$

Throughout $t \in (-r^2, r^2)$, choose numbers r_l^b, r_l^f ($l = 1, 2, \dots, L_\lambda$) so that they satisfy

$$\frac{\Delta \rho_l}{12} \int_{\{\rho_l^b\} \times \mathbb{S}^{d-1}} |\nabla u_\lambda(t, x)|^2 d\mathcal{H}_x^{d-1} = \int_{\rho_l^b - \Delta \rho_l / 12}^{\rho_l^b + \Delta \rho_l / 12} \rho^{d-1} d\rho \int_{\mathbb{S}^{d-1}} |\nabla u_\lambda(t, x)|^2 d\omega_{d-1},$$

$$\frac{\Delta \rho_l}{12} \int_{\{\rho_l^f\} \times \mathbb{S}^{d-1}} |\nabla u_\lambda(t, x)|^2 d\mathcal{H}_x^{d-1} = \int_{\rho_l^f - \Delta \rho_l / 12}^{\rho_l^f + \Delta \rho_l / 12} \rho^{d-1} d\rho \int_{\mathbb{S}^{d-1}} |\nabla u_\lambda(t, x)|^2 d\omega_{d-1}$$

and $r_1^f = \rho_1, r_0 = 0, r_{L_\lambda+1}^b = r$,

$$\tilde{r}_l = \begin{cases} 0 & (l = 0) \\ (1 - \epsilon_0^4)r & (l = 1) \\ (1 - \epsilon_0^4)r + C_2 \epsilon_0^4 r \sum_{j=1}^{l-1} \Delta \theta_j^3 & (l = 2, \dots, L_\lambda) \end{cases}$$

with $C_2 = \left(\sum_{l=1}^{L_\lambda-1} \Delta \theta_l^3 \right)^{-2}$.

Next introduce a mapping f_λ which is the solution to

$$\begin{cases} \nabla_{|x|} f_\lambda + \frac{r}{2(d-1)} \Delta_\tau f_\lambda = 0 & \text{in } [0, r) \times \mathbb{S}^{d-1} \\ f_\lambda = u_\lambda & \text{on } \{r\} \times \mathbb{S}^{d-1}. \end{cases} \quad (2.20)$$

Designate four sorts of annulus

$$\begin{aligned} T'_l &= [\rho_{l-1}, \rho_l) \times \mathbb{S}^{d-1}, \quad T_l^b = [r_l^b, r_l^f) \times \mathbb{S}^{d-1}, \\ T_l^f &= [r_l^f, r_{l+1}^b) \times \mathbb{S}^{d-1}, \quad T_l = T_l^b \cup T_l^f = [r_l^b, r_{l+1}^b) \times \mathbb{S}^{d-1} \\ & \quad (l = 1, 2, \dots, L_\lambda). \end{aligned} \quad (2.21)$$

A first step to prove a Hybrid type inequality is to inductively construct a certain support mappings $w'_{\lambda,l}$, $w_{\lambda,l}^b$ and $w_{\lambda,l}^f$ by making the best of our

support function f_λ : They are the solutions of

$$\begin{cases} -\Delta w'_{\lambda,l} = 0 & \text{in } T_l \\ w'_{\lambda,l}|_{\{\rho_{l-1}\} \times \mathbb{S}^{d-1}} = f_\lambda|_{\{\tilde{r}_{l-1}\} \times \mathbb{S}^{d-1}} & (l = 2, \dots, L_\lambda), \\ w'_{\lambda,l}|_{\{\rho_l\} \times \mathbb{S}^{d-1}} = f_\lambda|_{\{\tilde{r}_l\} \times \mathbb{S}^{d-1}} \end{cases} \quad (2.22)$$

$$\begin{cases} -\Delta w'_{\lambda,1} = 0 & \text{in } [0, r_1^f) \times \mathbb{S}^{d-1} \\ w'_{\lambda,1}|_{\{r_1^f\} \times \mathbb{S}^{d-1}} = f_\lambda|_{\{\tilde{r}_1\} \times \mathbb{S}^{d-1}}, \end{cases} \quad (2.23)$$

$$\begin{cases} -\Delta w_{\lambda,l}^b = 0 & \text{in } T_l^b \\ w_{\lambda,l}^b|_{\{r_l^b\} \times \mathbb{S}^{d-1}} = w'_{\lambda,l}|_{\{r_l^b\} \times \mathbb{S}^{d-1}} & (l = 2, \dots, L_\lambda), \\ w_{\lambda,l}^b|_{\{r_l^f\} \times \mathbb{S}^{d-1}} = w'_{\lambda,l}|_{\{r_l^f\} \times \mathbb{S}^{d-1}} \end{cases} \quad (2.24)$$

$$\{w_{\lambda,1}^b = w'_{\lambda,1}, \quad (2.25)$$

$$\begin{cases} -\Delta w_{\lambda,l}^f = 0 & \text{in } T_l^f \\ w_{\lambda,l}^f|_{\{r_l^f\} \times \mathbb{S}^{d-1}} = w'_{\lambda,l}|_{\{r_l^f\} \times \mathbb{S}^{d-1}} & (l = 2, \dots, L_\lambda) \\ w_{\lambda,l}^f|_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} = w'_{\lambda,l}|_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} \end{cases} \quad (2.26)$$

and then set mapping $w_{\lambda,l}$ by

$$w_{\lambda,l} = \begin{cases} w_{\lambda,l}^b & \text{in } [r_l^b, r_l^f) \times \mathbb{S}^{d-1} \\ w_{\lambda,l}^f & \text{in } [r_l^f, r_{l+1}^b) \times \mathbb{S}^{d-1} \end{cases} \quad (l = 2, \dots, L_\lambda), \quad (2.27)$$

$$w_{\lambda,1} = w_{\lambda,1}^f \quad \text{in } [0, r_1^f) \times \mathbb{S}^{d-1}. \quad (2.28)$$

We state a property on the mappings $w_{\lambda,l}$ used below.

Lemma 2.15 *The mappings $w_{\lambda,l}^b$ and $w_{\lambda,l}^f$ have the following property:*

$$\begin{aligned} & |x \cdot \nabla w_{\lambda,l}^b|^2 + |x \cdot \nabla w_{\lambda,l}^f|^2 \\ & \leq \frac{C}{\Delta r_l^{d-2}} \left(\int_{\tilde{r}_{l-1}}^{\tilde{r}_l} \rho^{d-1} d\rho \int_{\{\rho\} \times \mathbb{S}^{d-1} \cap B_{\Delta r/2}(\rho, x/\rho)} |\nabla_\rho f_\lambda|^2 d\mathcal{H}_y^{d-1} \right. \\ & \quad \left. + \Delta r_l \int_{\{\tilde{r}_l\} \times \mathbb{S}^{d-1} \cap B_{\Delta r/2}(\tilde{r}_l, x/\tilde{r}_l)} |\nabla_\tau f_\lambda|^2 d\mathcal{H}_y^{d-1} \right) \end{aligned} \quad (2.29)$$

holds for any point x in T_l .

Proof of Lemma 2.15.

The estimates for $x \cdot \nabla w_{\lambda,l}^b$ is performed by the straight-forward computation from the explicit formula, the mean value theorem for Laplace equation and the sub-harmonic estimates.

$$\begin{aligned} w_{\lambda,l}^b(x) &= w_{\lambda,l}^b(\rho, \omega_{d-1}/\rho) \\ &= \sum_{\substack{n=1, \\ \alpha \in N(n)}}^{\infty} a_n^{(\alpha)} \left(\frac{\rho}{r_l} \right)^n \phi_n^{(\alpha)}(\omega_{d-1}) + \sum_{\substack{n=1, \\ \alpha \in N(n)}}^{\infty} b_n^{(\alpha)} \left(\frac{r_{l-1}}{\rho} \right)^n \phi_n^{(\alpha)}(\omega_{d-1}) + a_0 \phi_0(\omega_{d-1}) \end{aligned}$$

with

$$\begin{aligned} a_n^{(\alpha)} &= \frac{f_{\lambda}^{n,(\alpha)}(t, \tilde{r}_{l-1}) \tau_l^n - f_{\lambda}^{n,(\alpha)}(t, \tilde{r}_l)}{\tau_l^{2n} - 1}, \\ b_n^{(\alpha)} &= \frac{f_{\lambda}^{n,(\alpha)}(t, \tilde{r}_l) \tau_l^n - f_{\lambda}^{n,(\alpha)}(t, \tilde{r}_{l-1})}{\tau_l^{2n} - 1}, \\ \tau_l &= \frac{r_{l-1}}{r_l}, \\ f_{\lambda}^{n,(\alpha)}(t, r) &= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} \langle f_{\lambda}(t, r, \omega_{d-1}), \phi_n^{(\alpha)}(\omega_{d-1}) \rangle d\omega_{d-1}, \end{aligned} \quad (2.30)$$

where $\{\phi_n^{(\alpha)}\}$ ($n = 0, 1, \dots; \alpha \in N(n)$) is a sequence of the independent hyper-spherical harmonics and $N(n)$ is the number of independent hyper-spherical harmonics with degree n . The estimate on $x \cdot \nabla w_{\lambda,l}^f$ is similar. \square

After the preparation above, we show the proof of Theorem 2.14.

Proof of Theorem 2.14.

Take the difference between (2.14) and $-\Delta w_{\lambda,l} = 0$ on T_l^b and T_l^f , multiplying it by $-2x \cdot \nabla(u_{\lambda} - w_{\lambda,l})$, integrate it on T_l^b and T_l^f and sum up it for l to verify

$$\begin{aligned} &-2 \sum_{l=1}^{L_{\lambda}} \int_{T_l} \left\langle \frac{\partial u_{\lambda}}{\partial t}, x \cdot \nabla(u_{\lambda} - w_{\lambda,l}) \right\rangle dx \\ &+ (d-2) \sum_{l=1}^{L_{\lambda}} \int_{T_l} |\nabla(u_{\lambda} - w_{\lambda,l})|^2 dx + \frac{d\lambda^{1-\kappa}}{2} \sum_{l=1}^{L_{\lambda}} \int_{T_l} (|u_{\lambda}|^2 - 1)^2 dx \\ &= -2\lambda^{1-\kappa} \sum_{l=1}^{L_{\lambda}} \int_{T_l} (|u_{\lambda}|^2 - 1) \langle u_{\lambda}, x \cdot \nabla w_{\lambda,l} \rangle dx \end{aligned}$$

$$\begin{aligned}
& - \sum_{l=1}^{L_\lambda-1} \left(r_{l+1}^b \int_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} - r_l^f \int_{\{r_l^f\} \times \mathbb{S}^{d-1}} \right) |\nabla_{|x|}(u_\lambda - w_{\lambda,l}^f)|^2 d\mathcal{H}_x^{d-1} \\
& - \sum_{l=1}^{L_\lambda-1} \left(r_l^f \int_{\{r_l^f\} \times \mathbb{S}^{d-1}} - r_l^b \int_{\{r_l^b\} \times \mathbb{S}^{d-1}} \right) |\nabla_{|x|}(u_\lambda - w_{\lambda,l}^b)|^2 d\mathcal{H}_x^{d-1} \\
& + \sum_{l=1}^{L_\lambda-1} \left(r_{l+1}^b \int_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} - r_l^b \int_{\{r_l^b\} \times \mathbb{S}^{d-1}} \right) |\nabla_\tau(u_\lambda - w_{\lambda,l}^b)|^2 d\mathcal{H}_x^{d-1} \\
& + \frac{\lambda^{1-\kappa}}{2} \sum_{l=1}^{L_\lambda-1} \left(r_{l+1}^b \int_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} - r_l^b \int_{\{r_l^b\} \times \mathbb{S}^{d-1}} \right) (|u_\lambda|^2 - 1)^2 d\mathcal{H}_x^{d-1} \\
& = \text{(I)} + \text{(II)} + \cdots + \text{(V)}. \tag{2.31}
\end{aligned}$$

From now on, we shall estimate the each term of the right-hand side in (2.31). First we estimate the first term (I): Choose a sequence of balls $\{B_{\Delta r_l}(x_{i_l})\}$ ($i_l \in I_l$) with

$$\begin{aligned}
T_l & \subset \cup_{i_l \in I_l} B_{2\Delta r_l/3}(x_{i_l}), \\
\cup_{i_l \in I_l} B_{\Delta r_l}(x_{i_l}) & \subset T_{l-1} \cup T_l \cup T_{l+1}.
\end{aligned}$$

Notice that

$$\begin{aligned}
& \lambda^{1-\kappa} \int_{B_{2\Delta r_l/3}(x_{i_l})} (1 - |u_\lambda|^2) dx \leq C \int_{B_{\Delta r_l}(x_{i_l})} \mathbf{e}_\lambda dx + \frac{C}{\Delta r_l} \int_{B_{\Delta r_l}(x_{i_l})} |\nabla u_\lambda| dx \\
& + \int_{B_{\Delta r_l}(x_{i_l})} \left| \frac{\partial u_\lambda}{\partial t} \right| dx.
\end{aligned}$$

We then obtain

$$\begin{aligned}
\text{(I)} & \leq \lambda^{1-\kappa} \sum_{l=2}^{L_\lambda-1} \sum_{i_l \in I_l} \int_{B_{2\Delta r_l/3}(x_{i_l})} (1 - |u_\lambda|^2) |x \cdot \nabla w_{\lambda,l}| dx \\
& + \lambda^{1-\kappa} \sum_{i_1 \in I_1} \int_{B_{2\Delta r_1/3}(x_{i_1})} (1 - |u_\lambda|^2) |x \cdot \nabla w_{\lambda,1}| dx \\
& + \lambda^{1-\kappa} \sum_{i_{L_\lambda} \in I_{L_\lambda}} \int_{B_{2\Delta r_{L_\lambda}/3}(x_{i_{L_\lambda}})} (1 - |u_\lambda|^2) |x \cdot \nabla w_{\lambda,L_\lambda}| dx
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{l=2}^{L_\lambda-1} \sum_{\mathbf{i}_l \in I_l B_{\Delta r_l}(x_{\mathbf{i}_l})} \int \mathbf{e}_\lambda \sup_{x \in B_{2\Delta r_l/3}(x_{\mathbf{i}_l})} |x \cdot \nabla w_{\lambda,l}| dx \\
&+ \sum_{l=2}^{L_\lambda-1} \sum_{\mathbf{i}_l \in I_l B_{\Delta r_l}(x_{\mathbf{i}_l})} \int \left| \frac{\partial u_\lambda}{\partial t} \right| \sup_{x \in B_{2\Delta r_l/3}(x_{\mathbf{i}_l})} |x \cdot \nabla w_{\lambda,l}| dx \\
&+ C \sum_{l=2}^{L_\lambda-1} \frac{1}{r \Delta r_l} \sum_{\mathbf{i}_l \in I_l B_{\Delta r_l}(x_{\mathbf{i}_l})} \int |\nabla u_\lambda| \sup_{x \in B_{2\Delta r_l/3}(x_{\mathbf{i}_l})} |x \cdot \nabla w_{\lambda,l}| dx \\
&+ C \sum_{\mathbf{i}_1 \in I_1 B_{\Delta r_1}(x_{\mathbf{i}_1})} \int \mathbf{e}_\lambda \sup_{x \in B_{2\Delta r_1/3}(x_{\mathbf{i}_1})} |x \cdot \nabla w_{\lambda,1}| dx \\
&+ C \sum_{\mathbf{i}_1 \in I_1 B_{\Delta r_1}(x_{\mathbf{i}_1})} \int \left| \frac{\partial u_\lambda}{\partial t} \right| \sup_{x \in B_{2\Delta r_1/3}(x_{\mathbf{i}_1})} |x \cdot \nabla w_{\lambda,1}| dx \\
&+ C \sum_{\mathbf{i}_1 \in I_1 B_{\Delta r_1}(x_{\mathbf{i}_1})} \int |\nabla u_\lambda| \sup_{x \in B_{2\Delta r_1/3}(x_{\mathbf{i}_1})} |x \cdot \nabla w_{\lambda,1}| dx \\
&+ C \lambda \Delta r_{L_\lambda} r^d \sup_{B_r \setminus B_{L_\lambda}^{\mathbf{i}}} |\nabla_{|x|} w_{\lambda,L_\lambda}^{\mathbf{i}}| \\
&\leq C \left(\int_{B_r \setminus B_{(1-\epsilon_0^4)r}} \mathbf{e}_\lambda dx \right)^{1/2} \left(\sum_{l=2}^{L_\lambda-1} \sum_{\mathbf{i}_l \in I_l B_{\Delta r_l}(x_{\mathbf{i}_l})} \int \mathbf{e}_\lambda dx \sup_{x \in B_{2\Delta r_l/3}(x_{\mathbf{i}_l})} |x \cdot \nabla w_{\lambda,l}|^2 \right)^{1/2} \\
&+ C \left(\int_{B_r \setminus B_{(1-\epsilon_0^4)r}} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 dx \right)^{1/2} \left(\sum_{l=2}^{L_\lambda-1} \sum_{\mathbf{i}_l \in I_l} \frac{\Delta r_l^d}{r^2} \sup_{x \in B_{2\Delta r_l/3}(x_{\mathbf{i}_l})} |x \cdot \nabla w_{\lambda,l}|^2 \right)^{1/2} \\
&+ \left(\int_{B_r \setminus B_{(1-\epsilon_0^4)r}} |\nabla u_\lambda|^2 dx \right)^{1/2} \left(\sum_{l=2}^{L_\lambda-1} \sum_{\mathbf{i}_l \in I_l} \Delta r_l^{d-2} \sup_{x \in B_{2\Delta r_l/3}(x_{\mathbf{i}_l})} |x \cdot \nabla w_{\lambda,l}|^2 \right)^{1/2} \\
&+ \left(\int_{B_{r_1}} \mathbf{e}_\lambda dx \right)^{1/2} \left(\int_{B_{r_1}} \mathbf{e}_\lambda dx \sup_{x \in B_{r_1}} |x \cdot \nabla w_{\lambda,1}|^2 \right)^{1/2} \\
&+ C \left(r^2 \int_{B_{r_1}} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 dx \right)^{1/2} \left(r^{d-2} \sup_{x \in B_{r_1}} |x \cdot \nabla w_{\lambda,1}|^2 \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
& + C \left(\int_{B_{r_1}} |\nabla u_\lambda|^2 dx \right)^{1/2} \left(r^{d-2} \sup_{x \in B_{r_1}} |x \cdot \nabla w_{\lambda,1}|^2 \right)^{1/2} \\
& + C \lambda^2 \Delta r_{L_\lambda} r^d.
\end{aligned}$$

By applying Corollary 2.7, Lemma 2.15 and the sub-harmonic estimate for $x \cdot \nabla w_{\lambda,l}$, we can proceed to our evaluation.

$$\begin{aligned}
\text{(I)} & \leq \frac{C}{\epsilon_0^2} \int_{B_r \setminus B_{(1-\epsilon_0^4)r}} \left(\mathbf{e}_\lambda + r^2 \left| \frac{\partial u_\lambda}{\partial t} \right|^2 \right) dx + C \epsilon_0^2 \int_{B_r \setminus B_{(1-\epsilon_0^4)r}} |\nabla_{|x|} f_\lambda|^2 dx \\
& + C \epsilon_0^2 \int_{B_{(1-\epsilon_0^4)r}} \left(\mathbf{e}_\lambda + \left| \frac{\partial u_\lambda}{\partial t} \right|^2 \right) dx \\
& + \frac{C}{\epsilon_0^2} \int_{B_{(1-\epsilon_0^4)r}} |\nabla_{|x|} f_\lambda|^2 dx + C \lambda^2 \Delta r_{L_\lambda} r^d,
\end{aligned}$$

by the choice of Δr_{L_λ} which follows to conclude

$$\begin{aligned}
\text{(I)} & \leq C \int_{B_r \setminus B_{r(1-\epsilon_0^4)}} \left(\mathbf{e}_\lambda + r^2 \left| \frac{\partial u_\lambda}{\partial t} \right|^2 \right) dx \\
& + C \left(\frac{1}{\epsilon_0^2} \int_{B_r \setminus B_{r(1-\epsilon_0^4)}} |\nabla_\tau u_\lambda|^2 dx + \epsilon_0^2 r \int_{\partial B_r} |\nabla_\tau u_\lambda|^2 d\mathcal{H}_x^{d-1} \right) \quad (2.32) \\
& + \frac{C(\epsilon_0)}{r} \int_{\partial B_r} |u_\lambda - a|^2 d\mathcal{H}_x^{d-1} + \frac{Cr^d}{\lambda}.
\end{aligned}$$

Successively this implies

$$\begin{aligned}
\text{(II)} + \text{(III)} & \leq -r \int_{\partial B_r} |\nabla_{|x|} u_\lambda|^2 d\mathcal{H}_x^{d-1} \\
& + 2 \sum_{l=1}^{L_\lambda-1} r_{l+1}^b \int_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} \langle \nabla_{|x|} u_\lambda, \nabla_{|x|} w_{\lambda,l}^f \rangle d\mathcal{H}_x^{d-1} \\
& - 2 \sum_{l=1}^{L_\lambda} r_l^f \int_{\{r_l^f\} \times \mathbb{S}^{d-1}} \langle \nabla_{|x|} u_\lambda, \nabla_{|x|} w_{\lambda,l}^f \rangle d\mathcal{H}_x^{d-1}
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{l=1}^{L_\lambda} r_l^f \int_{\{r_l^f\} \times \mathbb{S}^{d-1}} \langle \nabla_{|x|} u_\lambda, \nabla_{|x|} w_{\lambda,l}^b \rangle d\mathcal{H}_x^{d-1} \\
& - 2 \sum_{l=1}^{L_\lambda} r_l^b \int_{\{r_l^b\} \times \mathbb{S}^{d-1}} \langle \nabla_{|x|} u_\lambda, \nabla_{|x|} w_{\lambda,l}^f \rangle d\mathcal{H}_x^{d-1} \\
& + \sum_{l=1}^{L_\lambda} r_l^f \int_{\{r_l^f\} \times \mathbb{S}^{d-1}} |\nabla_{|x|} w_{\lambda,l}^f|^2 d\mathcal{H}_x^{d-1} + \sum_{l=1}^{L_\lambda} r_l^b \int_{\{r_l^b\} \times \mathbb{S}^{d-1}} |\nabla_{|x|} w_{\lambda,l}^b|^2 d\mathcal{H}_x^{d-1} \\
& \leq -r \int_{\partial B_r} |\nabla_{|x|} u_\lambda|^2 d\mathcal{H}_x^{d-1} \\
& + \sum_{l=1}^{L_\lambda-1} \Delta r_l \int_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} |\nabla_{|x|} u_\lambda|^2 d\mathcal{H}_x^{d-1} + \sum_{l=1}^{L_\lambda-1} \Delta r_l \int_{(\{r_l^b\} \cup \{r_l^f\}) \times \mathbb{S}^{d-1}} |\nabla_{|x|} u_\lambda|^2 d\mathcal{H}_x^{d-1} \\
& + \sum_{l=1}^{L_\lambda-1} \frac{Cr^2}{\Delta r_l} \int_{\{r_{l+1}^b\} \times \mathbb{S}^{d-1}} |\nabla_{|x|} w_{\lambda,l}^f|^2 d\mathcal{H}_x^{d-1} \\
& + \sum_{l=1}^{L_\lambda-1} \frac{Cr^2}{\Delta r_l} \int_{(\{r_l^b\} \cup \{r_l^f\}) \times \mathbb{S}^{d-1}} |\nabla_{|x|} w_{\lambda,l}^b|^2 d\mathcal{H}_x^{d-1}.
\end{aligned}$$

Using Lemma 2.15, from a definition on r_l^b and r_l^f , we arrive at

$$\begin{aligned}
\text{(II)} + \text{(III)} & \leq C \left(\frac{1}{\epsilon_0^2} \int_{B_r \setminus B_{(1-\epsilon_0^4)r}} |\nabla u_\lambda|^2 dx + \epsilon_0^2 r \int_{\partial B_r} |\nabla u_\lambda|^2 d\mathcal{H}_x^{d-1} \right) \\
& + \frac{C}{\epsilon_0^2 r} \int_{\partial B_r} |u_\lambda - a|^2 d\mathcal{H}_x^{d-1} + \frac{C(\epsilon_0, r)}{\sqrt{\lambda}}. \tag{2.33}
\end{aligned}$$

From construction on $w_{\lambda,l}$, we readily find that (IV) vanishes. Finally (V) becomes

$$\text{(V)} = \frac{r\lambda^{1-\kappa}}{2} \int_{\partial B_r} (|u_\lambda|^2 - 1)^2 d\mathcal{H}_x^{d-1}. \tag{2.34}$$

We also have the following estimate for the left-hand side in (2.31) called (L):

$$\begin{aligned} (L) \geq & \frac{d-2}{4} \int_{B_{(1-\epsilon_0^4)r}} \mathbf{e}_\lambda dx - C r^2 \int_{B_r} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 dx \\ & - \frac{C}{r} \int_{\partial B_r} |u_\lambda - a|^2 d\mathcal{H}_x^{d-1}. \end{aligned} \quad (2.35)$$

A substitution of (2.32), (2.33), (2.34) and (2.35) for (2.31), an integration of it with respect to $t \in (-r^2, r^2)$ verifies

$$\begin{aligned} \int_{P_{r/2}} \mathbf{e}_\lambda dz & \leq C \epsilon_0^2 \int_{P_r} \mathbf{e}_\lambda dz + C r^2 \int_{P_r} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 dz \\ & + \frac{C(\epsilon_0)}{r} \int_{-r^2}^{r^2} dt \int_{\partial B_r} |u_\lambda - a|^2 d\mathcal{H}_x^{d-1} \\ & + \int_{-r^2}^{r^2} \frac{r \lambda^{1-\kappa}}{2} dt \int_{\partial B_r} (|u_\lambda|^2 - 1)^2 d\mathcal{H}_x^{d-1}. \end{aligned} \quad (2.36)$$

Integrate (2.36) from $R/2$ to R with respect to r and divide it by R to obtain

$$\begin{aligned} \int_{P_{R/4}} \mathbf{e}_\lambda dz & \leq C \epsilon_0 \int_{P_R} \mathbf{e}_\lambda dz + \frac{C(\epsilon_0)}{R^2} \int_{P_R} |u_\lambda - a|^2 dz \\ & + C R^2 \int_{P_R} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 dz + \frac{C(R, \epsilon_0)}{\log \lambda}. \end{aligned} \quad (2.37)$$

To complete the proof, we attempt to cover finely any fixed parabolic cylinder $P_R(z_0)$ by a family of small parabolic cylinder whose diameter is $\epsilon_0 R$. For this purpose, set N_1 be $(2([1/\epsilon_0] + 1))^d$. We equivalently divide the parabolic cube $D_R(z_0) = (t_0 - R^2, t_0 + R^2) \times C_R(x_0)$ into small parabolic cubes: There is a finite sequence of small parabolic cubes $\{D_{\epsilon_0 R}(t_{p'}, x_{q'})\}$ ($p = 1, 2, \dots, 2([1/\epsilon_0^2] + 1)$; $q = 1, 2, \dots, N_1$) with

$$\begin{aligned} D_{\epsilon_0 R}(t_{p'}, x_{q'}) \cap D_{\epsilon_0 R}(t_{r'}, x_{s'}) & = \emptyset \text{ if } p' \neq r' \text{ or } q' \neq s', \\ D_R(z_0) & \subset \bigcup_{p=1}^{2[1/\epsilon_0^2]+2} \bigcup_{q=1}^{N_1} \text{“the closure of } D_{\epsilon_0 R}(t_{p'}, x_{q'})\text{”}. \end{aligned}$$

If we take a family of parabolic cylinders $\{P_{\epsilon_0 R}(t_p, x_q)\}$ whose centre (t_p, x_q) is located at all the centre and the vertex of each small parabolic cube $D_{\epsilon_0 R}(t_{p'}, x_{q'})$ above, it is fulfilled

$$P_R(z_0) \subset D_R(z_0) \subset \bigcup_{(t_p, x_q)} \text{“the closure of } P_{\epsilon_0 R}(t_p, x_q)\text{.”} \quad (2.38)$$

We must remark that the number of $P_{4\epsilon_0 R}(t_p, x_q)$ that includes any fixed point in $D_R(z_0)$ is bounded by d . Once we assign each element of $\{P_{\epsilon_0 R}(t_p, x_q)\}$ to $P_{R/4}$ in (2.37) and sum it up over all such a parabolic cylinder, we obtain

$$\begin{aligned} \sum_{p,q} \int_{P_{\epsilon_0 R}(t_p, x_q)} \mathbf{e}_\lambda dz &\leq C\epsilon_0^2 \sum_{p,q} \int_{P_{4\epsilon_0 R}(t_p, x_q)} \mathbf{e}_\lambda dz \\ &+ \frac{C(d, \epsilon_0)}{(\epsilon_0 R)^2} \sum_{p,q} \int_{P_{4\epsilon_0 R}(t_p, x_q)} |u_\lambda - a|^2 dz + C(d, \epsilon_0)(\epsilon_0 R)^2 \sum_{p,q} \int_{P_{4\epsilon_0 R}(t_p, x_q)} \left| \frac{\partial u_\lambda}{\partial t} \right|^2 dz \\ &+ \frac{C(R, \epsilon_0)}{\log \lambda}. \end{aligned} \quad (2.39)$$

Since the inclusion

$$\bigcup_{p,q} P_{4\epsilon_0 R}(t_p, x_q) \subset P_{3R/2}$$

holds, it implies that in view of (2.39) and (2.13) in Theorem 2.8, taking a positive number ϵ_0 so small with $C\epsilon_0 < 1$, we describe

$$\begin{aligned} \int_{P_R(z_0)} \mathbf{e}_\lambda dz &\leq \epsilon_0 \int_{P_{2R}(z_0)} \mathbf{e}_\lambda dz + \frac{C(\epsilon_0)}{R^2} \int_{P_{2R}(z_0)} |u_\lambda - a|^2 dz \\ &+ \frac{C(R, \epsilon_0)}{\log \lambda}, \end{aligned} \quad (2.40)$$

which thereby completes the proof. \square

3 WHHF

3.1 Existence

This chapter studies the existence and a partial regularity on WHHF. The existence theorem is a slight modification of Y.Chen [8]; See also L.C.Evans [15, p.48, 5.A.1] and J.Shatah [28]. First of all we mention convergence theorem directly derived from Theorem 2.5 in GLHF:

Theorem 3.1 (Convergence). *There exist a subsequence $\{u_{\lambda(\nu)}\}$ ($\nu = 1, 2, \dots$) of $\{u_\lambda\}$ ($\lambda > 0$) in $V(Q(T); \mathbb{S}^D)$ and a mapping $u \in V(Q(T); \mathbb{S}^D)$ such that the sequence of mappings $\{u_{\lambda(\nu)}\}$ ($\nu = 1, 2, \dots$) respectively converges weakly and weakly-* to a mapping u in $H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and $L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$. So does it strongly to the mapping u in $L^2(Q(T); \mathbb{S}^D)$ and point-wisely to it in almost all $z \in Q(T)$ as $\nu \nearrow \infty$.*

Theorem 3.1 enables us state the following existence theorem:

Theorem 3.2 (Existence). *The GLHF converges to a WHHF in $L^2(Q(T); \mathbb{R}^{D+1})$ as $\lambda \nearrow \infty$ (modulo a subsequence of λ).*

Remark 3.3 *We hereafter fix the subsequences $\{\lambda(\nu)\}$ ($\nu = 1, 2, \dots$) of $\{\lambda\}$ ($\lambda > 0$) chosen in Theorem 3.1*

3.2 Partial Regularity

The next section discusses a partial regularity on the WHHF obtained through the limit of the GLHF.

Definition 3.4 *Let $\{u_{\lambda(\nu)}\}$ ($\nu = 1, 2, \dots$) be the sequence selected above and set $\mathbf{e}_{\lambda(\nu)}$ the Ginzburg-Landau energy density $|\nabla u_{\lambda(\nu)}|^2/2 + \lambda(\nu)^{1-\kappa}(|u_{\lambda(\nu)}|^2 - 1)^2/4$. We then denotes $\overline{\mathcal{M}}$ by*

$$\overline{\mathcal{M}}(P_R(z_0)) = \limsup_{\lambda(\nu) \nearrow \infty} \frac{1}{R^d} \int_{P_R(z_0)} \mathbf{e}_{\lambda(\nu)} dz$$

where $P_R(z_0)$ is an arbitrary parabolic cylinder compactly contained in $Q(T)$.

Lemma 3.5 (Measured Hybrid Inequality). *Assume that the a sequence of GLHF $\{u_{\lambda(\nu)}\}$ ($\nu = 1, 2, \dots$), respectively converges weakly and weakly-* in $H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and*

$L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$ to a WHHF $u \in V(Q; \mathbb{S}^D)$ as $\lambda(\nu) \nearrow \infty$. Then take the pass to the limit $\lambda(\nu) \nearrow \infty$ in Theorem 2.14 to infer the following: For any positive number ϵ_0 , there exists a positive constant $C(\epsilon_0)$ satisfying $C(\epsilon_0) \nearrow \infty$ as $\epsilon_0 \searrow 0$ such that the inequality

$$\overline{\mathcal{M}}(P_R(z_0)) \leq \epsilon_0 \overline{\mathcal{M}}(P_{2R}(z_0)) + C(\epsilon_0) \int_{P_{2R}(z_0)} |u(z) - a(t)|^2 dz \quad (3.1)$$

holds for any time-variable L^2 -mapping $a(t)$ ($= (a^i(t))$ ($i = 1, 2, \dots, D+1$)) and any parabolic cylinder $P_{2R}(z_0)$ compactly contained in $Q(T)$.

In the similar manner as in L. Simon [29, Lemma 2, p.31], we can assert the following reverse Poincaré inequality.

Corollary 3.6 (Reverse Poincaré Inequality). *Let a mapping $a (= a(t))$ be a time variable mapping in $L^2((0, T); \mathbb{R}^{D+1})$. Then (3.1) implies the reverse Poincaré inequality*

$$R^{d+2} \overline{\mathcal{M}}(P_R(z_0)) \leq C \int_{P_{2R}(z_0)} |u(z) - a(t)|^2 dz \quad (3.2)$$

holds whenever $P_{2R}(z_0)$ is an arbitrary parabolic cylinder compactly contained in $Q(T)$.

By combining Corollary 3.6 with Theorem 2.8 and using Sobolev imbedding theorem and Poincaré inequality for the space variables, we can describe the following lemma. We refer the proof to Theorem 2.1 in M. Giaquinta and M. Struwe [20].

Lemma 3.7 *There exists a positive number q_0 greater than 1 such that differentials ∇u of the WHHF u belongs to $L_{loc}^{2q_0}(Q(T); \mathbb{R}^{d(D+1)})$ with*

$$\left(\int_{P_R(z_0)} |\nabla u(z)|^{2q_0} dz \right)^{1/q_0} \leq C \int_{P_{2R}(z_0)} |\nabla u(z)|^2 dz. \quad (3.3)$$

If we follow the result by Y. Chen and M. Struwe [11, Lemma 2.4], we can claim

Theorem 3.8 *For any positive number ϵ , set*

$$\mathbf{sing}(\epsilon) = \bigcap_{R>0} \{z_0 \in Q(T); \overline{\mathcal{M}}(P_R(z_0)) \geq \epsilon, P_R(z_0) \subset\subset Q(T)\}, \quad (3.4)$$

$$\mathbf{reg}(\epsilon) = Q(T) \setminus \mathbf{sing}. \quad (3.5)$$

Then there exist some positive number ϵ_0 and an increasing function $g(t)$ with $g(0) = 0$ and $g(t) = O(t \log(1/t)^{d+1})$ ($t \searrow 0$) such that if $z_0 \in \mathbf{reg}(\epsilon_0)$, that is for some positive number R_0 and positive integer λ_0 possibly depending on z_0 ,

$$\frac{1}{R_0^d} \int_{P_{g(R_0)}(z_0)} \mathbf{e}_\lambda(z) dz < \epsilon_0 \quad (3.6)$$

implies

$$\sup_{z \in P_{R_0}(z_0)} \mathbf{e}_\lambda(z) \leq \frac{C}{R_0^2}, \quad (3.7)$$

as long as any λ is more than or equal to λ_0 .

Definition 3.9 In the sequel, we respectively mean **sing** and **reg** by **sing**(ϵ_0) and **reg**(ϵ_0), respectively.

Lemma 3.10 Pick up any point $z_0 \in \mathbf{reg}$ and fix it. On the parabolic cylinder $P_{R_0/2}(z_0)$ which is the half size of the cylinder in (3.7), the inequality

$$|u_\lambda(t, x) - u_\lambda(s, x)| \leq \frac{C}{R_0} |t - s|^{1/2} \quad (3.8)$$

holds for any points t and s in $[t_0 - (R_0/2)^2, t_0 + (R_0/2)^2]$ and $x \in \overline{B}_{R_0/2}(x_0)$ with $z_0 = (t_0, x_0)$.

Proof of Corollary 3.10.

Assume $s < t$; Then combining Theorem 2.8 with Theorem 3.8 as $R = \sqrt{t - s}$, we infer

$$\begin{aligned} & \int_{B_{\sqrt{t-s}}(x)} |u_\lambda(t, y) - u_\lambda(s, y)| dy \\ & \leq |t - s|^{(d+2)/4} \left(\int_s^t d\tau \int_{B_{\sqrt{t-s}}(x)} \left| \frac{\partial u_\lambda}{\partial \tau} \right|^2 dy \right)^{1/2} \\ & \leq C |t - s|^{(d+2)/4} \left(\frac{1}{|t - s|} \int_{(s+t)/2 - (t-s)}^{(s+t)/2 + (t-s)} d\tau \int_{B_{2\sqrt{t-s}}(x)} \mathbf{e}_\lambda dy \right)^{1/2} \\ & \leq \frac{C}{R_0} |t - s|^{(d+1)/2}. \end{aligned} \quad (3.9)$$

By plying (3.9) and (3.7), we calculate the term of $|u_\lambda(t, x) - u_\lambda(s, x)|$:

$$\begin{aligned} & |u_\lambda(t, x) - u_\lambda(s, x)| \\ & \leq \left| u_\lambda(t, x) - \int_{B_{\sqrt{t-s}}(x)} u_\lambda(t, y) dy \right| + \left| u_\lambda(s, x) - \int_{B_{\sqrt{t-s}}(x)} u_\lambda(s, y) dy \right| \end{aligned}$$

$$\begin{aligned}
& + \int_{B_{\sqrt{t-s}}(x)} |u_\lambda(t, y) - u_\lambda(s, y)| dy \\
& \leq \int_{B_{\sqrt{t-s}}(x)} dy \int_0^{|y-x|} \left(\left| \frac{\partial u_\lambda}{\partial \rho} \left(t, x + \rho \frac{y-x}{|y-x|} \right) \right| + \left| \frac{\partial u_\lambda}{\partial \rho} \left(s, x + \rho \frac{y-x}{|y-x|} \right) \right| \right) d\rho \\
& + \frac{C}{R_0} |t-s|^{1/2} \leq \frac{3C}{R_0} |t-s|^{1/2}. \quad \square \quad (3.10)
\end{aligned}$$

Theorem 3.11 (Singular Set). *The set of **sing** is a relatively closed set and*

$$\mathcal{H}^{(d)}(\mathbf{sing}) = 0 \quad (3.11)$$

holds with respect to the parabolic metric.

Proof of Theorem 3.11.

sing is a relatively closed set. Indeed, if $z_0 \in \overline{\mathbf{sing}} \cap Q(T)$, some sequence $\{z_\mu\}$ ($\mu = 1, 2, \dots$) $\subset \mathbf{sing} \cap Q(T)$ satisfies $z_\mu \rightarrow z_0$ as $\mu \nearrow \infty$. More precisely for any positive number δ , there exists a positive integer μ_δ such that $d(z_\mu, z_0) < \delta$ holds for an arbitrary positive integer $\mu \geq \mu_\delta$. From definition on **sing**, for any $R > \delta$ and any points z_μ ($\mu = \mu_\delta, \mu_\delta + 1, \dots$) $\in \text{textbf{sing}} \cap Q(T)$, we obtain

$$\begin{aligned}
\epsilon_0 & \leq \frac{1}{(R-\delta)^d} \limsup_{\lambda(\nu) \nearrow \infty} \int_{P_{R-\delta}(z_\mu)} \mathbf{e}_{\lambda(\nu)}(z) dz \\
& \leq \frac{1}{(R-\delta)^d} \limsup_{\lambda(\nu) \nearrow \infty} \int_{P_R(z_0)} \mathbf{e}_{\lambda(\nu)}(z) dz. \quad (3.12)
\end{aligned}$$

By the arbitrariness of δ , passing to the limit of $\delta \searrow 0$, we can say $\overline{\mathbf{sing}} \cap Q(T) \subset \mathbf{sing} \cap Q(T)$, which provides us with our first assertion.

Next we estimate the size of **sing** in the d -dimensional Hausdorff measure with respect to the parabolic metric. Fix a positive integer n and a positive number R : Set a compact set $Q_n = [1/n^2, T - 1/n^2] \times \overline{B_{1-1/n}}(0)$ and let $\{P_{2R_k}(z_k)\}$ ($0 < 2R_k < \min(R, 1/(2n))$) be a cover of **sing**. The parabolic version of Vitali covering theorem shows that there is a disjoint finite subfamily $\{P_{2R_j}(z_j)\}$ ($j \in \mathcal{J}$) with

$$\mathbf{sing} \cap Q_n \subset \bigcup_{j \in \mathcal{J}} P_{2R_j}(z_j), \quad \epsilon_0 R_j^d \leq \limsup_{\lambda(\nu) \nearrow \infty} \int_{P_{R_j}(z_j)} \mathbf{e}_{\lambda(\nu)}(z) dz.$$

From Corollary 3.6, we infer

$$\begin{aligned} \epsilon_0 R_j^d &\leq \limsup_{\lambda(\nu) \nearrow \infty} \int_{P_{R_j}(z_j)} \mathbf{e}_{\lambda(\nu)}(z) dz \\ &\leq \frac{C}{R_j^2} \int_{P_{2R_j}(z_j)} |u(z) - u_{B_{2R_j}(x_j)}(t)|^2 dz \leq C \int_{P_{2R_j}(z_j)} |\nabla u(z)|^2 dz. \end{aligned}$$

Thus we obtain

$$\sum_{j=1}^J (20R_j)^d \leq C \int_{\cup_{j=1}^J P_{2R_j}(z_j)} |\nabla u(z)|^2 dz, \quad (3.13)$$

$$\text{The relation of } \sum_{j=1}^J (2R_j)^{d+2} \leq CR^2 \int_{Q(T)} |\nabla u(z)|^2 dz,$$

from (3.13) and the absolute continuity of the Lebesgue integration, concludes

$$\mathcal{H}^{(d)}(\mathbf{sing} \cap Q_n) \leq C \lim_{R \searrow 0} \sum_{j=1}^J (20R_j)^d = 0. \quad (3.14)$$

By $\lim_{n \rightarrow \infty} \mathcal{H}^{(d)}(\mathbf{sing} \cap Q_n) = \mathcal{H}^{(d)}(\mathbf{sing})$, we can deduce our assertion. \square

3.3 Compactness

We have seen in the previous section that the GLHF u_λ converges weakly to a WHHF u (modulo a subsequence of λ). Theorem 3 in L. C. Evans [15, p.39] expounds that the differentials of the GLHF, ∇u_λ does strongly to them of the WHHF, ∇u in $L^p(Q(T); \mathbb{R}^{d(D+1)})$ with $1 < p < 2$. But it doesn't suffice to prove that the WHHF u satisfies the monotonicity for the scaled energy, Corollary 2.13. Thus we demonstrate the strong convergency of $\{u_{\lambda(\nu)}\}$ ($\nu = 1, 2, \dots$) to a WHHF u in $H_{\text{loc}}^{1,2}$ -topology as $\lambda(\nu) \nearrow \infty$. The estimates of $\mathcal{H}^{(d)}(\mathbf{sing}) = 0$ in Theorem 3.11 plays a crucial role in the proof.

Theorem 3.12 (Strong Convergency of Gradients of WHHF). *For a suitable subsequence of $\{\lambda(\nu)\}$ still denoted by $\{\lambda(\nu)\}$ ($\nu = 1, 2, \dots$), a sequence of the gradients of the GLHF, $\{\nabla u_{\lambda(\nu)}\}$ ($\nu = 1, 2, \dots$) converges strongly to the gradients of the WHHF in $L_{\text{loc}}^2(Q(T); \mathbb{R}^{d(D+1)})$.*

Proof of Theorem 3.12.

Set $Q_n = [1/n^2, T - 1/n^2] \times \overline{B_{1-1/n}(0)}$ ($n = 1, 2, \dots$) and fix any compact sets $Q_n \subset\subset Q_{2n} \subset\subset Q(T)$ and any positive integer k . By means of $\mathcal{H}^{(d)}(\mathbf{sing}) = 0$, we first see that we can choose up two finite sets of cylinders $\{P_{r_{i,k}}(z_{i,k})\}$ and $\{P_{\rho_{j,k}}(z'_{j,k})\}$ ($i = 1, 2, \dots, I_k$; $j = 1, 2, \dots, J_k$) satisfy that

$$\mathbf{sing} \cap Q_{2n} \subset \bigcup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k}), \quad \sum_{i=1}^{I_k} r_{i,k}^d \leq \frac{1}{k}, \quad (3.15)$$

$$Q_{2n} \setminus \bigcup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k}) \subset \bigcup_{j=1}^{J_k} P_{\rho_{j,k}}(z'_{j,k}) \subset\subset Q(T) \quad (3.16)$$

$$\text{with } \overline{\mathcal{M}}(P_{g(\rho_{j,k})}(z'_{j,k})) < \epsilon_0 \quad (3.17)$$

where the function g is the positive function appeared in Theorem 3.8. In addition by a diagonal argument, we find that we can pick up a subsequence $\{\lambda(\nu(l))\}$ ($l = 1, 2, \dots$) of $\{\lambda(\nu)\}$ ($\nu = 1, 2, \dots$) with the following properties:

- (i) $\lambda(\nu(l)) \geq l$,
- (ii) for any positive integer k , $|u_{\lambda(\nu(l))}(z) - u(z)| \leq 1/k$ holds on any point $z \in Q_{2n} \setminus \bigcup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})$ whenever $l \geq k$.

Indeed from (3.17), it follows that for some $\lambda(\nu_{j,k})$ depending on j and k ,

$$\frac{1}{g(2\rho_{j,k})^d} \int_{P_{g(2\rho_{j,k})}(z'_{j,k})} \mathbf{e}_{\lambda(\nu)}(z) dz < \epsilon_0$$

holds if $\lambda(\nu)$ is more than or equal to $\lambda(\nu_{j,k})$. Then Theorem 3.8 and Lemma 3.10 can read

$$|u_{\lambda(\nu)}(z_1) - u_{\lambda(\nu)}(z_2)| \leq \frac{C J_k}{\rho_{j,k}} d(z_1, z_2) \quad (3.18)$$

for any points z_1 and $z_2 \in Q_{2n} \setminus \bigcup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})$.

When we set $\bar{\nu}_k = \max_{j=1,2,\dots,J_k} \nu_{j,k}$ and $\underline{\rho}_k = \min_{j=1,2,\dots,J_k} \rho_{j,k}$ in (3.18), Ascoli-Arzelà's theorem claims that there exists a subsequence $\{\lambda(\nu(l))\}$ ($l = 1, 2, \dots$) of $\{\lambda(\nu)\}$ ($\nu = 1, 2, \dots$) such that for some $\nu(k)$ more than or equal to $\bar{\nu}_k$ and k if $\nu(l) \geq \nu(k)$,

$$|u_{\lambda(\nu(l))}(z) - u(z)| \leq \frac{1}{k} \quad (3.19)$$

holds on any point $z \in Q_{2n} \setminus \bigcup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})$, where the mapping u is the WHHF constructed in Theorem 3.2.

Next we do exactly the same procedure above for $k + 1$ instead of k . So we can select a number $\lambda(\nu(k + 1))$ from $\{\lambda(\nu(l))\}$ ($l = 1, 2, \dots$) more than or equal to $\lambda(\nu(k))$ and $k + 1$ respectively satisfying

$$\begin{aligned} |u_{\lambda(\nu(k+1))}(z) - u(z)| &\leq \frac{1}{k+1}, \\ |u_{\lambda(\nu(k+1))}(z) - u(z)| &\leq \frac{1}{k}, \end{aligned} \quad (3.20)$$

holds on any point $z \in Q_{2n} \setminus \bigcup_{i=1}^{I_{k+1}} P_{r_{i,k+1}}(z_{i,k+1})$ and $Q_{2n} \setminus \bigcup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})$.

By an induction, we can choose a subsequence $\{\lambda(\nu(k))\}$ ($k = 1, 2, \dots$) of $\{\lambda\}$ ($\lambda > 0$) satisfying monotone nondecreasing with respect to k , and

$$\begin{aligned} \lambda(\nu(l)) &\geq \max(\lambda(\nu(k)), l), \\ |u_{\lambda(\nu(l))}(z) - u(z)| &\leq \frac{1}{k} \quad \text{on } Q_{2n} \setminus \bigcup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k}) \end{aligned}$$

holds for any integers k and l with $k \leq l$.

An adaptation of $\lambda(\nu(k))$ to $\lambda(k)$ shows our claim.

We shall confirm that the sequence of $\{\nabla u_{\lambda(k)}\}$ ($k = 1, 2, \dots$) does converge *strongly* to the gradient of the WHHF u appeared in Theorem 3.2 in $L^2_{\text{loc}}(Q_{2n}; \mathbb{R}^{d(D+1)})$ as $k \nearrow \infty$. To this end, fix any positive integer k and let $\lambda(l)$ be more than or equal to $\lambda(k)$. Take the difference between (2.14) in $\lambda = \lambda(l)$ and (1.5),

$$\begin{aligned} &\int_{Q_{2n}} \left\langle \frac{\partial}{\partial t}(u_{\lambda(l)} - u), \phi \right\rangle dz + \int_{Q_{2n}} \langle \nabla(u_{\lambda(l)} - u), \nabla \phi \rangle dz \\ &= - \int_{Q_{2n}} |\nabla u|^2 \langle u, \phi \rangle dz + \int_{Q_{2n}} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u_{\lambda(l)}, \phi \rangle dz \end{aligned} \quad (3.21)$$

for a map $\phi \in C_0^\infty(Q_{2n}; \mathbb{R}^{D+1})$. Choose a smooth function η_1 satisfying

$$\eta_1(z) = \begin{cases} 1 & \text{on } Q_n, \\ 0 & \text{off } Q_{2n} \end{cases} \quad (3.22)$$

with $0 \leq \eta_1 \leq 1$, $|\nabla \eta_1| \leq 2n$, $|\Delta \eta_1| + |\partial \eta_1 / \partial t| \leq 16n^2$.

We substitute a smooth approximation of $(u_{\lambda(l)} - u)\eta_1$ for ϕ in (3.21). After passing to the limit, we obtain

$$\int_{Q_{2n}} |\nabla(u_{\lambda(l)} - u)|^2 \eta_1 dz = \frac{1}{2} \int_{Q_{2n}} |u_{\lambda(l)} - u|^2 \left(\frac{\partial \eta_1}{\partial t} + \Delta \eta_1 \right) dz$$

$$\begin{aligned}
& - \int_{Q_{2n}} |\nabla u|^2 \langle u, u_{\lambda(l)} - u \rangle \eta_1 dz \\
& + \int_{Q_{2n}} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u_{\lambda(l)}, u_{\lambda(l)} - u \rangle \eta_1 dz.
\end{aligned} \tag{3.23}$$

From now on, we compute to pass to the limit $\lambda(l) \nearrow \infty$ on the right-hand side in (3.23). By using the strong convergency of $u_{\lambda(l)}$ in $L^2(Q(T); \mathbb{R}^{D+1})$, i.e. Theorem 3.1, we can calculate the first term as follows:

$$\limsup_{\lambda(l) \nearrow \infty} \int_{Q_{2n}} |u_{\lambda(l)} - u|^2 \left(\frac{\partial \eta_1}{\partial t} + \triangle \eta_1 \right) dz = 0. \tag{3.24}$$

Next we estimate the second term on the right-hand side: noting $|u_{\lambda(l)}| \leq 1$, thanks to the dominated Lebesgue convergence theorem and Theorem 3.1, we obtain

$$\limsup_{\lambda(l) \nearrow \infty} \int_{Q_{2n}} |\nabla u|^2 |u_{\lambda(l)} - u| dz = 0. \tag{3.25}$$

Finally we asses the third term. We decompose it into

$$\begin{aligned}
& \int_{Q_{2n}} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u_{\lambda(l)}, u_{\lambda(l)} - u \rangle \eta_1 dz \\
& = \int_{\cup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u_{\lambda(l)}, u_{\lambda(l)} - u \rangle \eta_1 dz \\
& + \int_{Q_{2n} \setminus \cup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u_{\lambda(l)}, u_{\lambda(l)} - u \rangle \eta_1 dz.
\end{aligned} \tag{3.26}$$

We majorize the first term in (3.26) as follows: Recall a way of choosing a sequence of cylinder $\{P_{r_{i,k}}(z_{i,k})\}$ ($i = 1, 2, \dots, I_k$) with (3.15) for any positive integer k and set a certain smooth cut off functions $\phi_{i,k}$ with $0 \leq \phi_{i,k} \leq 1$, $|\nabla \phi_{i,k}| \leq 2/r_{i,k}$, $|\triangle \phi_{i,k}| \leq 4/r_{i,k}^2$ and $|\partial \phi_{i,k}/\partial t| \leq 2/r_{i,k}^2$ with

$$\phi_{i,k} = \begin{cases} 1 & \text{in } P_{r_{i,k}}(z_{i,k}), \\ 0 & \text{outside } P_{2r_{i,k}}(z_{i,k}). \end{cases}$$

Multiply (2.14) in $\lambda = \lambda(l)$ by $u_{\lambda(l)} \phi_{i,k}$ and integrate it on $P_{r_{i,k}}(z_{i,k})$ to observe

$$\int_{P_{r_{i,k}}(z_{i,k})} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) dz$$

$$\leq \frac{C}{r_{i,k}^2} \int_{P_{2r_{i,k}}(z_{i,k})} (1 - |u_{\lambda(l)}|^2) dz + C \int_{P_{2r_{i,k}}(z_{i,k})} \mathbf{e}_{\lambda(l)} dz. \quad (3.27)$$

Set $z_{i,k} = (t_{i,k}, x_{i,k})$ and recall $d_n = d(Q_n, \partial Q(T)) (= 1/n)$; Thus using Theorem 2.5 and Corollary 2.13, the second term in (3.27) can be evaluated

$$\begin{aligned} \int_{P_{r_{i,k}}(z_{i,k})} \mathbf{e}_{\lambda(l)} dz &= \int_{t_{i,k}+20r_{i,k}^2/3-8r_{i,k}^2/3}^{t_{i,k}+20r_{i,k}^2/3-32r_{i,k}^2/3} dt \int_{B_{r_{i,k}}(x_{i,k})} \mathbf{e}_{\lambda(l)} dx \\ &\leq \frac{Cr_{i,k}^d}{r_{i,k}^d} \int_{t_{i,k}+20r_{i,k}^2/3-32r_{i,k}^2/3}^{t_{i,k}+20r_{i,k}^2/3-8r_{i,k}^2/3} dt \int_{B_{r_{i,k}}(x_{i,k})} \mathbf{e}_{\lambda(l)} \exp\left(\frac{|x - x_{i,k}|^2}{4(t - (t_{i,k} + 20r_{i,k}^2/3))}\right) dx \\ &\leq Cr_{i,k}^d \left(\frac{1}{d_n^d} \int_{t_{i,k}+20r_{i,k}^2/3-d_n^2}^{t_{i,k}+20r_{i,k}^2/3-(d_n/2)^2} dt \int_{\mathbb{B}^d} \mathbf{e}_{\lambda(l)} dx + Cr_{i,k}^2 \right) \\ &\leq Cr_{i,k}^d. \end{aligned} \quad (3.28)$$

Then from (3.26), (3.27) and (3.28), we obtain

$$\begin{aligned} &\int_{\cup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u, u_{\lambda(l)} - u \rangle \eta_1 dz \\ &\leq C \sum_{i=1}^{I_k} \int_{P_{r_{i,k}}(z_{i,k})} \mathbf{e}_{\lambda}(z) dz + \sum_{i=1}^{I_k} \frac{C}{r_{i,k}^2} \int_{P_{r_{i,k}}(z_{i,k})} (1 - |u_{\lambda(l)}|^2) dz \\ &\leq C \sum_{i=1}^{I_k} r_{i,k}^d \leq \frac{C}{k}. \end{aligned} \quad (3.29)$$

On the other hand, since $|u_{\lambda(l)} - u| \leq 1/k$ on $Q_{2n} \setminus \cup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})$, likewise (3.27), the second term of (3.26) becomes

$$\begin{aligned} &\int_{Q_{2n} \setminus \cup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u_{\lambda(l)}, u_{\lambda(l)} - u \rangle \eta_1 dz \\ &\leq \frac{C}{k} \int_{Q_{2n} \setminus \cup_{i=1}^{I_k} P_{r_{i,k}}(z_{i,k})} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \eta_1 dz \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{k} \int_{Q_{2n}} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \eta_1 dz \\
&\leq \frac{C}{k} \left(\frac{C}{d_n^2} \int_{Q(T)} (1 - |u_{\lambda(l)}|^2) dz + T \int_{\mathbb{B}^d} |\nabla u_0|^2 dx \right) \quad (3.30)
\end{aligned}$$

as long as $\lambda(l) \geq \lambda(k)$. Taking supremum limit in (3.29) and (3.30), we deduce

$$\limsup_{\lambda(l) \nearrow \infty} \int_{Q_{2n}} \lambda(l)^{1-\kappa} (1 - |u_{\lambda(l)}|^2) \langle u_{\lambda(l)}, u_{\lambda(l)} - u \rangle \eta_1 dz \leq \frac{C}{k}. \quad (3.31)$$

Since k is an arbitrary positive integer, our conclusion follows from (3.23), (3.24), (3.25) and (3.31). \square

4 Proof of Main Theorems

By making the best of a few ingredients and properties on the WHHF and the GLHF, this chapter establishes Theorem 1.2 and Theorem 1.4 in Chapter 1.

4.1 Proof of Theorem 1.2

Let $\{\lambda(k)\}$ ($k = 1, 2, \dots$) be a subsequence of λ be chosen in Theorem 3.12. Theorem 3.2 tells us that the subsequence of the GLHF $\{u_{\lambda(k)}\}$ ($k = 1, 2, \dots$) converges to a WHHF u in $L^2(Q(T); \mathbb{R}^{D+1})$.

Next we discuss a partial regularity on the WHHF constructed above: On account of Theorem 2.5 and Theorem 3.12, we obtain

$$\mathbf{sing} = \bigcap_{R>0} \left\{ z_0 \in Q(T); \frac{1}{2R^d} \int_{P_R(z_0)} |\nabla u|^2 dz \geq \epsilon_0, P_R(z_0) \subset\subset Q(T) \right\} \quad (4.1)$$

$$\mathbf{reg} = Q(T) \setminus \mathbf{sing}, \quad (4.2)$$

where a number ϵ_0 is a positive constant appeared in Theorem 3.8. From Theorem 3.11, we see that \mathbf{sing} is relatively closed. First of all, we measure the size of \mathbf{sing} . Recall that by Lemma 3.7, the differentials ∇u of the WHHF belongs to $L_{\text{loc}}^{2q_0}(Q(T); \mathbb{R}^{d(D+1)})$ for some positive number q_0 greater than 1. So accordingly the inclusion

$$\mathbf{sing} \subset \bigcap_{R>0} \left\{ z_0 \in Q(T); \frac{1}{R^{d-2(q_0-1)}} \int_{P_R(z_0)} |\nabla u|^{2q_0} dz \geq (C\epsilon_0)^{q_0}, P_R(z_0) \subset\subset Q(T) \right\}$$

enjoys

$$\mathcal{H}^{d-2(q_0-1)}(\mathbf{sing}) < \infty. \quad (4.3)$$

Next we show that if $z_0 \in \mathbf{reg}$, the WHHF u is smooth on some parabolic cylinder $P_{R_0/4}(z_0)$ by plying Ladyžhenskaya, O. A., Solonnikov, V. A., Ural'ceva, N. N, [23]. Invoke Theorem 3.8 and Lemma 3.10 to arrive at

$$|\nabla u(z)| \leq \frac{C}{R_0} \quad \text{and} \quad |u(t, x) - u(s, x)| \leq \frac{C}{R_0} |t - s|^{1/2} \quad (4.4)$$

on any point z , (t, x) and $(s, x) \in P_{R_0/2}(z_0)$ for some positive number R_0 . We prepare a smooth cut-off function η_{R_0} given by

$$\eta_{R_0}(z) = \begin{cases} 1 & \text{in } P_{R_0/4}(z_0), \\ 0 & \text{off } P_{R_0/2}(z_0) \end{cases} \quad (4.5)$$

with $0 \leq \eta_{R_0} \leq 1$. Then the mapping $u\eta_{R_0}$ satisfies the following system:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) (u\eta_{R_0}) &= |\nabla u|^2 u\eta_{R_0} \\ &\quad - 2\langle \nabla \eta_{R_0}, \nabla u \rangle - u\Delta \eta_{R_0} \quad \text{in } P_{R_0/2}(z_0). \end{aligned} \quad (4.6)$$

By applying Ladyžhenskaya, O. A., Solonnikov, V. A., Ural'ceva, N. N. [23, p.341, Theorem 9.1], we know that the mapping of $u\eta_{R_0}$ belongs to $H^{2,q}(P_{R_0/2}(z_0))$ for any number q greater than 1 because $|\nabla u|^2 u\eta_{R_0} - 2\langle \nabla \eta_{R_0}, \nabla u \rangle - u\Delta \eta_{R_0}$ is bounded on $P_{R_0/2}(z_0)$. Then employ [23, p.80, Lemma 3.3] to verify $\nabla u \in C^{\alpha_0}(P_{R_0/4}(z_0))$ where α is any arbitrary positive number less than 1. So by having Schauder estimate at our disposal, it is shown $u \in C^{2+\alpha}(P_{R_0/4}(z_0))$. For Schauder estimate, we refer to [23, p.320, Theorem 5.2]. In the light of a boot strap argument, we finally get the smoothness of u on $P_{R_0/4}(z_0)$.

We readily see that the WHHF u satisfies a global energy inequality (i), a monotonicity for the scaled energy (ii) and a reverse Poincaré inequality (iii). In fact, the first and third inequalities are established by using Theorem 3.1 and taking the limit inferior of $\lambda(k) \nearrow \infty$ in (2.7) in Theorem 2.5 and (3.2) in Corollary 3.6. While the second is proved as follows: For a point $z_0 = (t_0, x_0)$, set $d_0 = \text{dist}(x_0, \partial \mathbb{B}^d)$. Recall Corollary 2.13 and note

$$\begin{aligned} & \frac{1}{R_2^d} \int_{t_0 - (2R_2)^2}^{t_0 - R_2^2} dt \int_{\mathbb{B}^d} \mathbf{e}_{\lambda(k)} \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right) dx \\ & \leq \frac{1}{R_2^d} \int_{t_0 - (2R_2)^2}^{t_0 - R_2^2} dt \int_{B_{1-d_0}(0)} \mathbf{e}_{\lambda(k)} \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right) dx \end{aligned}$$

$$+ \frac{1}{R_2^d} \int_{t_0-(2R_2)^2}^{t_0-R_2^2} dt \int_{\mathbb{B}^d \setminus B_{1-d_0}(0)} \mathbf{e}_{\lambda(k)} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) dx. \quad (4.7)$$

We pass to the limit $\lambda(k) \nearrow \infty$ in above (4.7): By using Theorem 3.12 for the first term and Theorem 2.5 for the second term, we obtain

$$\begin{aligned} & \frac{1}{R_2^d} \limsup_{\lambda(k) \nearrow \infty} \int_{t_0-(2R_2)^2}^{t_0-R_2^2} dt \int_{\mathbb{B}^d} \mathbf{e}_{\lambda(k)} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) dx \\ & \leq \frac{1}{R_2^d} \limsup_{\lambda(k) \nearrow \infty} \int_{t_0-(2R_2)^2}^{t_0-R_2^2} dt \int_{B_{1-d_0}(0)} \mathbf{e}_{\lambda(k)} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) dx \\ & + \frac{1}{R_2^d} \limsup_{\lambda(k) \nearrow \infty} \int_{t_0-(2R_2)^2}^{t_0-R_2^2} dt \int_{\mathbb{B}^d \setminus B_{1-d_0}(0)} \mathbf{e}_{\lambda(k)} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) dx \\ & \leq \frac{1}{R_2^d} \int_{t_0-(2R_2)^2}^{t_0-R_2^2} dt \int_{B_{1-d_0}(0)} |\nabla u|^2 \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right) dx \\ & + C(d_0) R_2 \int_{\mathbb{B}^d} |\nabla u_0|^2 dx, \end{aligned} \quad (4.8)$$

which assert (1.15). \square

4.2 Proof of Theorem 1.4

First, we prove that the limiting mapping u is a WHHF and it satisfies (i) on $Q(T)$. Set a positive number h sufficiently small; From Theorem 2.5, we infer

$$\begin{aligned} & \int_{T_1-h}^{T_1+h} dt_1 \int_{T_2-h}^{T_2+h} dt_2 \int_{t_1}^{t_2} dt \int_{\mathbb{B}^d} \left| \frac{\partial u}{\partial t}(z) \right|^2 dx \\ & + \int_{T_2-h}^{T_2+h} dt \int_{\mathbb{B}^d} \mathbf{e}_{\lambda}(t) dx \leq \int_{T_1-h}^{T_1+h} dt \int_{\mathbb{B}^d} \mathbf{e}_{\lambda}(t) dx \end{aligned} \quad (4.9)$$

holds for any positive numbers T_1 and T_2 with $0 < T_1 \leq T_2 < T$. Our sending $\lambda \nearrow \infty$ of (4.9), in which Theorem 2.5 and Theorem 3.12 are implemented,

we infer on the same T_1 and T_2 ,

$$\begin{aligned} & \int_{T_1+h}^{T_2-h} dt \int_{\mathbb{B}^d} \left| \frac{\partial u}{\partial t}(z) \right|^2 dx \\ & + \frac{1}{2} \int_{T_2-h}^{T_2+h} dt \int_{\mathbb{B}^d} |\nabla u(t)|^2 dx \leq \int_{T_1-h}^{T_1+h} dt \int_{\mathbb{B}^d} |\nabla u(t)|^2 dx. \end{aligned} \quad (4.10)$$

Thus we possibly pass to the limit $h \searrow 0$ at any Lebesgue points T_1 and T_2 with $0 < T_1 \leq T_2 < T$ to conclude

$$\int_{T_1}^{T_2} dt \int_{\mathbb{B}^d} \left| \frac{\partial u}{\partial t}(z) \right|^2 dx + \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u(T_2)|^2 dx \leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u(T_1)|^2 dx. \quad (4.11)$$

Recalling (2.14) in Remark 2.10, by (4.11), Theorem 3.1 and Theorem 3.2, we construct a WHHF $u \in V(Q(T); \mathbb{S}^D)$ with

$$\begin{aligned} & \int_0^T dt \int_{\mathbb{B}^d} \left| \frac{\partial u}{\partial t}(z) \right|^2 dx + \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u(T)|^2 dx \leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u_0|^2 dx, \\ & \int_{Q(T)} \left\langle \frac{\partial u}{\partial t}, \phi \right\rangle dz + \int_{Q(T)} \langle \nabla u, \nabla \phi \rangle dz = \int_{Q(T)} |\nabla u|^2 \langle u, \phi \rangle dz \end{aligned}$$

for any map $\phi \in C^0(0, T; C_0^\infty(\mathbb{B}^d; \mathbb{R}^{D+1}))$.

By the same procedure above on $(T, 2T]$, we can extend the WHHF in $Q(T)$ to it in $Q(2T)$ and a repeat argument permits us to comprise a WHHF $u \in V(Q(\infty))$.

Next we prove the constancy property. A combination of Theorem 2.6 with Theorem 3.1 and Theorem 3.2 yields

$$\int_{\mathbb{B}^d} |\nabla u(t)|^2 dx \leq 2e^{-(d-2)t} \int_{\mathbb{B}^d} |\nabla u_0|^2 dx \quad \text{for all } t \in (0, \infty). \quad (4.12)$$

By the mapping $u = \text{the constant}$ on $\partial \mathbb{B}^d$, we thus deduce that the WHHF u converges strongly to it as $t \nearrow \infty$ in $L^2(\mathbb{B}^d)$. \square

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